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# Mathematical Reviews

*Edited by*

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# Mathematical Reviews

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## HISTORY

**Gandz, Solomon. Studies in Babylonian mathematics. I. Indeterminate analysis in Babylonian mathematics.** Osiris 8, 12–40 (1948).

The author discusses four geometrical problems which were previously published and investigated by the reviewer and by Thureau-Dangin. All of these problems contain corrupt or otherwise difficult passages which the present author suggests should be explained as indeterminate problems, similar to problems found in Diophantus.

*O. Neugebauer* (Providence, R. I.).

**van der Waerden, B. L. Die Arithmetik der Pythagoreer. II. Die Theorie des Irrationalen.** Math. Ann. 120, 676–700 (1949).

[For part I see the same vol., 127–153 (1948); these Rev. 9, 483.] This paper contains an analysis of the relationship and origin of books VIII, IX and X of Euclid's Elements. The author reaches the conclusion that there existed two different approaches to the theory of irrational magnitudes, one related to Archytas and the other ascribed to Theaitetos. The individual steps in the development of the theory are presented and the role of Theodoros is clarified.

*O. Neugebauer* (Providence, R. I.).

**Junge, Gustav. Flächenanlegung und Pentagramm. Ein Beitrag zur Entwicklungs-Geschichte von Euklids Elementen.** Osiris 8, 316–345 (1 plate) (1948).

The author intends to demonstrate that no oriental influence need be assumed for the origin of Greek "geometric algebra." To this end he shows that the problem of "application of areas" can be derived from the problem of the section in extreme and mean ratio. This problem is in turn connected with the construction of the pentagram and thus finally with the regular solids. *O. Neugebauer.*

***Heath, Thomas. Mathematics in Aristotle.** Oxford, at the Clarendon Press, 1949. xiv+291 pp. $5.50.

In 1904 J. L. Heiberg published a paper "Mathematisches zu Aristoteles" [Abh. Geschichte Math. Wiss. 18, 1–49 (1904)]. Heiberg's main interest was concentrated on the early development of works of the type of Euclid's "Elements" as reflected in the writings of Aristotle. At the same time he considered Byzantine commentaries concerning mathematical passages in Aristotle. The present work is of a different character, supplementing Heiberg's work. Arranged essentially according to the different Aristotelean works a practically complete collection of mathematical passages is attempted, including mechanics and some astronomy. All passages are translated and short comments are added. Unfortunately the material quoted by Aristotle is not very far reaching. Perhaps the most interesting section is a locus problem in connection with the optics of the rainbow. *O. Neugebauer* (Providence, R. I.).

**Thorndike, Lynn. Some little known astronomical and mathematical manuscripts.** Osiris 8, 41–72 (1948).

This is a list of about fifty treatises on various subjects contained in unpublished Latin mediaeval manuscripts from European libraries. *O. Neugebauer* (Providence, R. I.).

**Kamalamma, K. N. On the solution of a problem on indeterminate equations occurring in the Sulva-Sutras.** Bull. Calcutta Math. Soc. 40, 140–144 (1948).

The construction of altars for the religious ceremonies of Vedic times was of great importance to the Hindus. Detailed directions for the construction of such altars and for the attendant ceremonies are given in the Sulva Sutras but, where mathematical processes are involved, only brief statements of results are given. This paper formulates the construction data of specific types of altars as pairs of simultaneous equations with $n$ unknowns ($n=4$, 8, 10) and obtains solutions as sets of positive integers. It is stated that the solutions given in the Sulva Sutras are included among the solutions found by the methods of this paper.

*E. B. Allen* (Troy, N. Y.).

**Balagangadharan, K. A consolidated list of Hindu mathematical works.** Math. Student 15 (1947), 55–70 (1949).

A bibliography of Hindu mathematical and astronomical texts would be of great value for all scholars interested in this field of historical research. Unfortunately, the present list cannot satisfy this need. Its references are vague and dates of editions are not given. Classical works like the edition of the Pança-Siddhāntika by Thibaut and Dvivedi [Benares, 1889] are not mentioned. Modern bibliographical works like M. B. Emenau's A Union List of Printed Indic Texts, and Translations in American Libraries [Amer. Oriental Series, v. 7, New Haven, 1935] should have been consulted. No references are given as to the source of dates, even where they are controversial. *O. Neugebauer.*

**Loria, Gino. Perfectionnements, évolution, métamorphoses du concept de "coordonnées." Contribution à l'histoire de la géométrie analytique.** Osiris 8, 218–288 (1948).

***Pledge, H. T. Science Since 1500. A Short History of Mathematics, Physics, Chemistry, Biology.** Philosophical Library, New York, N. Y., 1949. 359 pp. (15 plates). $5.00.

First published by the Stationery Office, London, 1939; reprinted with minor corrections in 1940.

***Struik, Dirk J. Yankee Science in the Making.** Little, Brown and Company, Boston, Mass., 1948. xiii+430 pp. $5.00.

**Masotti, Arnaldo. Commemorazione di Umberto Cisotti.** Rend. Sem. Mat. Fis. Milano 18 (1947), i+1–35 (2 plates) (1948).

**Blaschke, Wilhelm. Galilei und Kepler.** Hamburger Math. Einzelschr. **39**, 17 pp. (1943).

**Vijayaraghavan, T. Obituary: G. H. Hardy.** Math. Student **15** (1947), 121–122 (1949).

**Efimov, N. V., Lopšic, A. M., and Raševskiĭ, P. K. Beniamin Fëdorovič Kagan (on his eightieth anniversary).** Uspehi Matem. Nauk (N.S.) **4**, no. 2(30), 5–14 (1 plate) (1949). (Russian)

**Whittaker, Edmund. Laplace.** Amer. Math. Monthly **56**, 369–372 (1949).

**Whittaker, Edmund. Laplace.** Math. Gaz. **33**, 1–12 (2 plates) (1949).

**Lenoble, Robert. A propos du tricentenaire de la mort de Mersenne.** Arch. Internat. Hist. Sci. (N.S.) **2(28)**, 583–597 (1949).

**Ślebodziński, W. Władysław Nikliborc et son oeuvre scientifique.** Colloquium Math. **1**, 322–330 (1948).

**Pelseneer, Jean. Obituary: Paul Ver Eecke.** Osiris **8**, 5–11 (1948).

## FOUNDATIONS

**Martin, R. M. A note on nominalism and recursive functions.** J. Symbolic Logic **14**, 27–31 (1949).

The author points out that the system described in a previous paper [same J. **8**, 1–23 (1943); these Rev. **4**, 182] resembles a system of W. V. Quine [same J. **12**, 74–84 (1947); these Rev. **9**, 261]. Like Quine's system, the author's is nominalistic in the sense that it is a functional calculus of first order in which the variables take only individual values, not functional values. It does not, however, satisfy Quine's severe finiteness restriction. After making some corrections to his former paper, he develops his system further by deriving in it the theory of ancestrals. He also shows that every function of natural numbers which is general recursive in the sense of Kleene may be defined in his system.

*O. Frink* (State College, Pa.).

**v. Wright, G. H. On the idea of logical truth. I.** Soc. Sci. Fenn. Comment. Phys.-Math. **14**, no. 4, 20 pp. (1948).

This is an exposition, in ordinary language with only rudimentary symbolism, of the method of tautologies in logic. There are three sections entitled "the logic of propositions," "the non-quantified logic of properties," and "the quantified logic of properties." The first two of these expound the method of truth tables in its usual form. The third gives, in effect, a decision process for those formulas of the lower predicate calculus in which only one place predicates occur, there are no propositional variables and no free individual variables, and the quantifications do not overlap. This is accomplished by exhibiting any such formula as a truth function of "existence constituents," each of which is of the form $(Ex)P(x)$, where $P(x)$ is a conjunction of the basic properties or their negations; there are a finite number of these existence constituents, and the given formula is identically true if and only if it is a tautology with respect to them. Of course, solutions of the decision problem for the predicate calculus of one variable are well known [see, e.g., Hilbert and Bernays, Grundlagen der Mathematik, v. 1, Springer, Berlin, 1934, p. 200, which gives references; Wajsberg, Monatsh. Math. Phys. **40**, 113–126 (1933)]; but the clear, nontechnical presentation here will doubtless be helpful to students. *H. B. Curry.*

**Goodman, Nelson. The logical simplicity of predicates.** J. Symbolic Logic **14**, 32–41 (1949).

This paper is related to an earlier one by the author [same J. **8**, 107–121 (1943); these Rev. **5**, 86], in which for a fully Platonistic logic a way of determining the relative simplicity of different sets of extralogical primitives was proposed, and to a recent paper by N. Goodman and W. V. Quine [same J. **12**, 105–122 (1947); these Rev. **9**, 262], in which the possibility of a purely nominalistic logic was shown. To an $n$-place predicate a complexity value $2n-1$ is assigned; in order to determine the complexity value of a basis, first the primitive predicates that are stipulated to be applicable are consolidated into a single one; then the sum is taken of the complexity values of all resulting predicates, applicable or inapplicable. In special cases, however, the calculated complexity value may have to be lowered on account of additional information available as to symmetry or other characteristics which increase the simplicity of the predicates involved. *E. W. Beth* (Amsterdam).

**Chauvin, André. Structures logiques.** C. R. Acad. Sci. Paris **228**, 1085–1087 (1949).

Given a logical system with (essentially) a simplified theory of types, the usual rule of substitution for universally quantified variables, and a formulation of the integers within the lowest type, the author shows that the reasoning of Gödel's proof of his first incompleteness theorem [Monatsh. Math. Phys. **38**, 173–198 (1931)] goes through provided one merely assumes the existence of a numeration of formulas such that a certain numerical relation has a representation in the system. This generalization can hardly fail to be apparent to anyone who understands Gödel's original paper; it differs from the generalization given by Gödel in his lectures at the Institute for Advanced Study in 1934 in that the properties which Gödel deduces from recursiveness are part of the hypothesis. *H. B. Curry* (State College, Pa.).

**Chauvin, André. Généralisation du théorème de Gödel.** C. R. Acad. Sci. Paris **228**, 1179–1180 (1949).

The Gödel theorem [cf. the preceding review] shows the undecidability, within a certain logical system, of a proposition constructed from a certain numerical relation $R(x, y)$. Let $\psi(x)$ be a numerical function defined in the system. Then, under certain broad restrictions, the author shows the undecidability of the proposition similarly constructed from $R(\psi(x), \psi(y))$. This gives a whole class of undecidable propositions. At the end the author raises two questions: (1) whether some of the unsolved problems of mathematics are reducible to propositions of this sort, and hence are undecidable, and (2) whether all undecidable propositions are of this form. (The author credits some suggestions along these lines to a book in Hungarian by R. Péter [Playing with the infinite, Mathematics for outsiders, Dante, Budapest, 1944; for a review of this book see J. Symbolic Logic **13**, 141–142 (1948)].)

*H. B. Curry* (State College, Pa.).

**Nelson, David. Constructible falsity.** J. Symbolic Logic **14**, 16–26 (1949).

This paper continues papers by S. C. Kleene [same J. **10**, 109–124 (1945); these Rev. **7**, 406] and D. Nelson [Trans. Amer. Math. Soc. **61**, 307–368 (1947); these Rev. **10**, 3]. With respect to a symbolic object-language containing no other variables than those for natural numbers, the notions of $P$-realization and $N$-realization are introduced. The natural number $a$ $P$-realizes the elementary formula $E$ if $a=0$ and $E$ is true; $a$ $N$-realizes $E$ if $a$ is any natural number and $E$ is false. For complex formulas the notion of realization is defined by induction. For instance $a$ $P$-realizes $B\supset C$, if $a$ is the Gödel number of a partial recursive function $\varphi$ such that, for every $b$ which $P$-realizes $B$, $\varphi(b)$ $P$-realizes $C$; $a$ $N$-realizes $B\supset C$ if $a=2^b3^c$ and $b$ $P$-realizes $B$ and $c$ $N$-realizes $C$. To establish the relation with Kleene's notion of realizability, to any formula $A$ there are connected by simple transformations two negation-free formulas $A'$ and $A''$ such that $a$ realizes $A$ if and only if $a$ $P$-realizes $A''$, and $a$ $P$-realizes $A$ if and only if $a$ realizes $A'$. A formal system $N$ is constructed such that every formula deducible in $N$ is $P$-realizable. In $N$ neither $A\vee\neg A$ nor $\neg(A\&\neg A)$ can be provable for all choices of $A$. By a method analogous to that by which Kleene extended his system $S$ to $S'$, $N$ is extended to $N'$ which is divergent from both the classical and the intuitionistic number theories. The adjunction of either $A\vee\neg A$ or $\neg(A\&\neg A)$ as an axiom scheme to $N'$ would result in an inconsistent system. *A. Heyting.*

**Fitch, Frederic B. Intuitionistic modal logic with quantifiers.** Portugaliae Math. **7**, 113–118 (1948).

This paper is concerned with a system of first-level predicate calculus obtained by adding to the primitives of the Heyting calculus the notions of logical possibility and logical necessity, for which the author uses the symbols '$\Diamond$' and '$\Box$,' respectively. The system differs from the usual systems of modal logic in that the nonmodal sentential connectives obey only the laws of the intuitionistic logic; moreover, $\Diamond$ and $\Box$ are not mutually interdefinable. A set of axioms for the system is given, based on the one rule of modus ponens for material implication. [Actually, the author gives two additional primitive rules, which assert that if $A$ is an axiom, then $\Box A$ and $(\alpha)A$ are axioms; but these rules could of course be avoided altogether, by simply taking three times as many axioms.] A set of proved rules is then derived, in the spirit of Gentzen and Jaśkowski, which is equivalent to the given axiom system.

No attempt is made to extend to this system the interpretation of the Heyting calculus given by Kleene, or the interpretation of ordinary modal logic given by McKinsey. (Such a task, it seems likely, would be a formidable undertaking.) Thus it is not clear exactly what the intuitive interpretation is intended to be. In this connection it is perhaps worthwhile to remark that no formula of the form $\sim\Diamond A$ is provable in the system; but whether the author intended his system to have this peculiarity I do not know. In rule XVIII, the last letter should be 'C' instead of 'B.'

*J. C. C. McKinsey* (Stillwater, Okla.).

**Fitch, Frederic B. The Heine-Borel theorem in extended basic logic.** J. Symbolic Logic **14**, 9–15 (1949).

A presumably consistent theory of real numbers was presented by the author in a previous paper [same J. **13**, 95–106 (1948); these Rev. **9**, 559]. In this theory a form of the least upper bound theorem is provable. In the present paper a form of the Heine-Borel theorem is proved on the basis of this theory. A class $S$ of ordered pairs of real numbers covers a pair of real numbers $(a, b)$ if for every real number $x$ such that $a\leqq x\leqq b$ there exists a pair $(c, d)$ such that $c<x<d$. The theorem then reads that, if a pair of real numbers $(a, b)$ is covered by a class $S$ of pairs of real numbers, then $(a, b)$ is also covered by a finite subclass of $S$. The proof appears to be of interest in view of the very restricted nature of the logic in which it is carried out.

*R. M. Martin* (Philadelphia, Pa.).

**Lorenzen, Paul. Gleichheit und Ungleichheit in der Arithmetik.** Math. Nachr. **1**, 337–341 (1948).

Der Verfasser führt für die Ziffern I, II, III, $\cdots$ die Relation "$=$" ein durch die Postulate (i) $a=a$, (ii) $ac=c\rightarrow\lambda$ und (iii) $c=ac\rightarrow\lambda$, wo $ac$ die Nebeneinanderstellung von $a$ und $c$ andeutet und $\lambda$ "das Falsche" bedeutet. Er definiert $(a\neq b)=_d(a=b\rightarrow\lambda)$. Dann beweist er mittels vollständiger Induktion, dass "$=$" eine klassenbildende Relation ist, dass $(\alpha)$ $a=b$, $a\neq b\rightarrow\lambda$, $(\beta)$ $a=b\vee a\neq b$ und dass $(\gamma)$ $a=b$, $\varphi(a)\rightarrow\varphi(b)$, wo $\varphi(a)$ eine der Formen $cae=c$, $c=dae$, $dae\neq c$, $c\neq dae$ hat. *A. Heyting* (Amsterdam).

**Borgers, Alfons. The natural number.** Simon Stevin **26**, 32–64, 65–73 (1949). (Dutch)

This is essentially a critical commentary on Dedekind's "Was sind und was sollen die Zahlen" [Vieweg, Braunschweig, 1888]. The author explains and discusses Dedekind's set-theoretic ideas and stresses their importance. From the standpoint of modern foundations about the most interesting point is a discussion, at the end, of the recursive definition of a sum, in which the definitions of Dedekind, Peano [as revised in Formulario Matematico, v. 5, Turin, 1905, pp. 30–31] and Kalmár [as given by E. Landau, Grundlagen der Analysis, Akademische Verlagsgesellschaft, Leipzig, 1930] are compared. The author argues that Kalmár's definition is not superior to Peano's, that none of the three definitions is completely satisfactory, and that here we encounter one of the most difficult problems of modern foundations [with this last, the reviewer disagrees; he believes the matter can easily be settled by the methods of Hilbert and Bernays, Grundlagen der Mathematik, v. 1, Springer, Berlin, 1934, in particular, pp. 286–294].

*H. B. Curry* (State College, Pa.).

**Kavun, N. I. A development of the theory of real numbers by the method of A. N. Kolmogorov.** Uspehi Matem. Nauk (N.S.) **2**, no. 5(21), 199–229 (1947). (Russian)

A. N. Kolmogorov [same journal (N.S.) **1**, no. 1(11), 217–219 (1946)] proposed the following definition of the concept of positive real number (only the nonnegative integers being presupposed). A positive real number is a function $\varphi(n)$ defined for all positive integers, whose values are nonnegative integers, and which possesses the following two properties: (1) for all positive integers $k$ and $n$, $\varphi(n)=[\varphi(kn)/k]$; (2) for every positive integer $n$ there exists a positive integer $k$ such that $\varphi(kn)>k\varphi(n)$. The function $\varphi(n)$, considered as a real number, is denoted by $\varphi$. (Post factum $\varphi(n)$ is the greatest integer $m$ such that $m<n\varphi$.) The fundamental concepts of arithmetic are introduced by the following three definitions: (i) $\varphi<\psi$ if, for at least one positive integer $n$, $\varphi(n)<\psi(n)$; (ii) $\varphi+\psi=\chi$ if $\chi(n)=\max_k[\{\varphi(kn)+\psi(kn)\}/k]$, where $k$ ranges over all positive integers; (iii) $\varphi\cdot\psi=\chi$, if

$$\chi(n)=\max_{k,k'}[\varphi(kn)\cdot\psi(k'n)/kk'],$$

where $k$ and $k'$ range over all positive integers. (In the above, $[x]$ denotes the greatest integer not less than $x$.) In the paper reviewed here a detailed proof is given of the fact that the set of real numbers obtained by this definition is isomorphic to that defined by any of the well-known methods. *A. Heyting* (Amsterdam).

**Neves Real, Luiz. On the definition of multiplication in the additive group of real numbers.** Gaz. Mat., Lisboa 10, no. 39, 4–5 (1949). (Portuguese)

An automorphism of the ordered group of reals has the trivial form, and hence the ordered group structure determines the product. *L. Nachbin* (Chicago, Ill.).

**Wang, Hao. On Zermelo's and von Neumann's axioms for set theory.** Proc. Nat. Acad. Sci. U. S. A. 35, 150–155 (1949).

In von Neumann's system of axioms for set theory, as revised by Bernays and Gödel ($N$) new variables $a, b, c, \cdots$ can be introduced which take as values only elements (classes capable of being members of other classes). In a formal way this is done by the definition-rules: (a) $\varphi a$ stands for $(x)((\exists y)(x\varepsilon y)\supset\varphi x)$; $(\exists a)\varphi a$ stands for $(\exists x)((\exists y)(x\varepsilon y).\varphi x)$. Then Zermelo's system as revised by Fränkel and Skolem ($Z$) is contained in $N$ in the following sense. If in a theorem in $Z$ the variables are considered as element variables of $N$, there results a theorem in $N$. By the introduction of new variables $X, Y, \cdots$, which are correlated with the general variables of $N$, $Z$ may be extended to a system $ZZ$, which is equivalent to $N$. Expressions such as $x\varepsilon Y$ are considered as atomic formulae in $ZZ$, while $X\varepsilon Y$ and $X\varepsilon y$ are introduced by definition-rules. Here $X\varepsilon Y$ stands for $(\exists z)(z=X.z\varepsilon Y)$, and analogously. One new axiom and one new rule have to be added to $Z$. The axiom asserts that $\xi=\eta\supset(\xi\varepsilon\zeta\equiv\eta\varepsilon\zeta)$, where $\xi, \eta, \zeta$ can be variables of both types. The rule states that in any formula containing free occurrences of $X$, $\hat{x}\varphi$ may be substituted for these occurrences, where $x$ is free in $\varphi$ and $\varphi$ contains no large variables. *A. Heyting.*

***Weyl, Hermann. Philosophy of Mathematics and Natural Science. Revised and Augmented English Edition Based on a Translation by Olaf Helmer.** Princeton University Press, Princeton, N. J., 1949. x+311 pp. $5.00.

The main portion of this introductory commentary is a translation (somewhat revised) of the article "Philosophie der Mathematik und Naturwissenschaft" contributed by Weyl to the Handbuch der Philosophie [Oldenbourg, München, 1926]. Part I outlines the principles of logic, the development of the number concept, and the invariants of geometry. Part II is concerned with (a) the mathematical, philosophical, and psychological aspects of space-time, (b) the making of measurements and the formation of concepts and theories, (c) various theories of matter and (d) the philosophical problems related to the idea of cause. Recent developments are treated in six appendices. (A) An exposition of Gödel's results concerning undecidable propositions and consistency, and attempts to avoid the classical antinomies. (B) The combinatorial basis of genetics. (C) The nature and properties of quantum states. (D) The quantum character of chemical valence. (E) Mutation and the philosophical problems of biology. (F) An essay on constancy and change. The historical development of many of these ideas is given in some detail. *C. C. Torrance.*

**Destouches-Février, P. Contradiction et complémentarité.** Synthèse 7, 173–182 (1949).

**Beth, E. W. Analyse sémantique des théories physiques.** Synthèse 7, 206–207 (1949).

**Freudenthal, H. Sur le besoin d'une logique matérielle.** Synthèse 7, 337–345 (1949).

**Huffer, E. Die Auffassung der Scholastik über die geometrischen Axiome.** Synthèse 7, 346–364 (1949).

## ALGEBRA

**Denk, Franz, und Haupt, Otto. Über gewöhnliche und signierte Permutationen.** J. Reine Angew. Math. 186, 170–183 (1945).

This note collects material on ordinary and signed permutations needed in Haupt's studies of singularities of certain $n$-dimensional space curves. The ordinary permutations appear as the lines connecting points 1 to $n$ running in the same direction on two straight lines, or as the wires connecting one set of $n$ terminals with another. In any such wire connection the wires are grouped in bundles [Büschel] such that all wires in one bundle cross each other, and each wire in any bundle (except the leftmost) does not cross at least one wire of any bundle to the left of it. The number of such bundles is the length of the permutation. For signed permutations, the second set of terminals is doubled by regarding each terminal as affected with a + or − sign; the number of (monomial) substitutions becomes $2^n\cdot n!$ Each signed permutation is shown to be coordinate with a partial set of the wires of the ordinary permutation, and hence also has a length. If $A(n, x)$ and $\bar{A}(n, x)$ denote the numbers of ordinary and signed permutations of length $x$, it is shown that $A(n, 1)=A(n, n)=1$, $A(n, n-1)=(n-1)^2$, $\bar{A}(n, 0)=n!$, $\bar{A}(n, n-1)=(n-1)^2+n^2$, $\bar{A}(n, n)=1$ and

$$\bar{A}(n, 1)=\sum_1^n \binom{n}{k}\binom{n}{k}(n-k)!$$

and complete enumerations of both kinds of numbers are given for $n=2, 3, 4$. The general determination of these numbers, which appears to be difficult, is left open.

*J. Riordan* (New York, N. Y.).

**Sprague, R. Über ein Anordnungsproblem.** Math. Ann. 121, 52–53 (1949).

Given a deck of $n$ distinct cards and a shuffle consisting of a cut and an interleaving of the resulting two parts, both the size of the cut and mode of interleaving being arbitrary, what is the least number of such shuffles to go from one card arrangement to another? The same problem appears in the classification of trains where the cars of a train may be assigned to one of two spurs each capable of receiving a full train and are reassembled by taking all cars on one spur first; this is the inverse of the shuffle. Any arrangement (permutation) is given a class number $k$, which is the number of times it is necessary to read it, going from left to right, to obtain the elements in natural order, e.g.,

$n(n-1)\cdots 1$ has class number $n$. The problem then becomes: what is the shortest expression of a permutation of class $k$ in factors of class 2? It is shown that $x$, the least number of such factors, is given by $2^{x-1}<k\leqq 2^x$. More generally, the least number of factors of class $j$ is given by $j^{x-1}<k\leqq j^x$. *J. Riordan* (New York, N. Y.).

**S[rinivasiengar], C. N., Gupta, Hansraj, Chakrabarty, S. C., Venkatachaliengar, K., and Tiruvenkatachar, V. R. On Kemmer's identity in combinatory functions.** Math. Student **15** (1947), 93–100 (1949).

Various proofs and extensions of an identity in binomial coefficients used by N. Kemmer [Proc. Cambridge Philos. Soc. **39**, 189–196 (1943); these Rev. **5**, 225] and running essentially as follows:

$$\binom{2r}{r}=\sum 2^{r-2p}\binom{r}{r-p}\binom{r-p}{p}.$$

*J. Riordan* (New York, N. Y.).

**Wasastjerna, Jarl A. Atomic arrangements with given number of neighbours.** Soc. Sci. Fenn. Comment. Phys.-Math. **14**, no. 3, 4 pp. (1948).

The following simple formula, arrived at by rough and ready argument, is proposed for the number of ways of arranging a large number $N$ of particles, $pN$ of which are $A$ and $qN$ are $B$ $(q=1-p)$, on a regular assembly which permits every particle $z$ geometrically identical nearest neighbors and such that the numbers of $AA$, $AB$, $BA$ and $BB$ neighbors are as $\kappa:\lambda:\mu:\nu$:

$$g(N,p,z,\kappa)=\{p^{\nu}q^{\mu}/\kappa^{\kappa}\lambda^{\lambda}\mu^{\mu}\nu^{\nu}\}^{N}.$$

This differs from the approximation given by Chang [Proc. Cambridge Philos. Soc. **35**, 265–292 (1939)] in being free of $z$ but is said to agree with the numerical results for $z=3$ of Chang and Ho [Proc. Roy. Soc. London. Ser. A. **180**, 345–365 (1942); these Rev. **3**, 320]. *J. Riordan.*

**Shannon, Claude E. The synthesis of two-terminal switching circuits.** Bell System Tech. J. **28**, 59–98 (1949).

In a previous paper [Trans. Amer. Inst. Elec. Engrs. **57**, 713–723 (1938)] the author has shown how Boolean algebra may be used to synthesize and simplify relay switching circuits. In this paper he extends these results and derives bounds for $\lambda(n)$, the minimum total number of relay contacts, and $\mu(n)$, the minimum of the largest number of contacts on any one relay, necessary to realize all possible switching functions of $n$ relays. Of the sixteen principal theorems, the following are samples. For all $n$, $\lambda(n)<2^{n+3}/n$; for every $\epsilon>0$ and $n$ sufficiently large,

$$(1-\epsilon)2^n/n<\lambda(n)<2^{n+2}/n;$$

for all $n$, $\mu(n)\leqq 2^{n+2}/n^2$.

Functions occurring in practice often have a high degree of symmetry. For these it is shown that the numbers given by $\lambda(n)$ and $\mu(n)$ are too large. For example, any switching function of $n$ relays, symmetric in $n-2$ of them, can be realized with $n^2-n+2$ or fewer contacts. All possible cases with four or fewer relays are worked out. It is shown that the number of two-terminal networks with $k$ or fewer branches is less than $(6k)^k$. *O. Frink.*

**Franckx, Ed. Sur le nombre de racines communes à deux équations algébriques.** Bull. Soc. Roy. Sci. Liége **16**, 145–148 (1947).

This paper indicates an elementary process for obtaining the number of common roots of two algebraic equations in one unknown. Write $x^i F(x)=q_i(x)f(x)+r_i(x)$, $i=0,1,\cdots$, where the degrees of the polynomials $F(x), f(x)$ are $m,n,m>n$, and the degrees of the $r_i(x)$ are less than $n$. If the greatest common divisor of $F(x)$, $f(x)$ is of degree $n-l$, there exist $l+1$ constants $\lambda_0,\lambda_1,\cdots,\lambda_l$, of which the last is not zero, such that the polynomial $\lambda_0 r_0(x)+\lambda_1 r_1(x)+\cdots+\lambda_l r_l(x)$ is identically zero. If $l$ is the smallest integer for which this is true, $F(x)=0$, $f(x)=0$ have $n-l$ common roots. Write $r_i(x)\equiv\sum_{j=0}^{n-1}\alpha_{ij}x^j$ and form the matrix $M\equiv((\alpha_{ij}))$, $i,j=0,\cdots,n-1$. If $p$ is the characteristic of $M$, the number of roots common to $F(x)=0$, $f(x)=0$ is equal to $n-p$. The equation which gives the common roots is $\sum_{k=0}^{p-1}\mu_k r_k(x)=0$, where the $\mu_k$ are constants determined by the equations $\sum_{k=0}^{p-1}\mu_k\alpha_{k,n-t}$, $t=p$, $p=1,\cdots,i$. *E. Frank.*

**Parodi, Maurice. Sur la détermination d'une limite supérieure de la partie réelle des racines de l'équation aux fréquences propres d'un réseau électrique.** C. R. Acad. Sci. Paris **228**, 1400–1402 (1949).

A procedure is developed for determining an upper bound for the real parts of the roots of the frequency equation $|l_{ij}z^2+r_{ij}z+s_{ij}|=0$ connected with an electrical network. Since the matrices $(l_{ij})$, $(r_{ij})$, $(s_{ij})$ are assumed symmetric and positive definite the roots have negative real parts. The method is partly based on a theorem by Ostrowski [Bull. Sci. Math. (2) **61**, 19–32 (1937)]. A simpler method can be used if the constants have the further property that $l_{ii}>\sum_{j\neq i}|l_{ij}|$, $r_{ii}>\sum_{j\neq i}|r_{ij}|$, $s_{ii}>\sum_{j\neq i}|s_{ij}|$.

*O. Todd-Taussky* (Washington, D. C.).

**Parodi, Maurice. Sur une application d'un théorème de M. J. Hadamard.** Bull. Sci. Math. (2) **72**, 136–138 (1948).

This is a discussion of sufficient conditions for the determinantal equation $|a_{ik}+z\delta_{ik}|=0$ to have no roots with positive real parts [see C. R. Acad. Sci. Paris **228**, 51–52, 807–808, 1198–1200 (1949); these Rev. **10**, 501; for related work also see S. Gerschgorin, Bull. Acad. Sci. URSS. Cl. Sci. Math. Nat. [Izvestiya Akad. Nauk SSSR] **1931**, 749–754; A. Brauer, Duke Math. J. **13**, 387–395 (1946); O. Taussky, ibid. **15**, 1043–1044 (1948); these Rev. **8**, 192; **10**, 501]. *O. Todd-Taussky* (Washington, D. C.).

**Parodi, M. Remarque sur la stabilité.** J. Phys. Radium (8) **10**, 200–201 (1949).

Cf. the preceding review. *R. P. Boas, Jr.*

**Hirsch, K. A. A note on Vandermonde's determinant.** J. London Math. Soc. **24**, 144–145 (1949).

Let $\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_n$ be elements of a commutative ring with unit element; let

$$\alpha_i^{(k)}=(\alpha_i+\beta_1)(\alpha_i+\beta_2)\cdots(\alpha_i+\beta_k);$$

and let $W_k$, $k=0,1,\cdots,n$, be the $n$th order determinant defined by

$$W_k=|1\alpha_i^{(1)}\alpha_i^{(2)}\cdots\alpha_i^{(k-1)}\alpha_i^{(k+1)}\cdots\alpha_i^{(n)}|.$$

The author proves that

$$W_k=V_n\left[a_{n-k}\cdot 1+a_{n-k-1}b_1+a_{n-k-2}\begin{vmatrix}-b_1 & b_2\\ 1 & -b_1\end{vmatrix}+\cdots\right],$$

where $V_n$ is the Vandermonde determinant $|1\alpha_i\alpha_i^2\cdots\alpha_i^{n-1}|$, where $a_0=1,a_1,\cdots,a_n$ are the elementary symmetric functions of $\alpha_1,\alpha_2,\cdots,\alpha_n$, and where $b_i$ is the $i$th elementary symmetric function formed from $\beta_1,\beta_2,\cdots,\beta_{k+1}$. If

$$\beta_1=\beta_2=\cdots=\beta_n=0,$$

this result reduces to the following well-known evaluation of the Vandermonde determinants with "missing" $k$th powers:

$$|1\alpha_i \cdots \alpha_i^{k-1}\alpha_i^{k+1} \cdots \alpha_i^n| = V_n \cdot a_{n-k}$$

[see Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, v. 2, Springer, Berlin, 1925, p. 99].

*G. B. Price* (Lawrence, Kan.).

**Walls, Nancy. A note on an identity of Jacobi's.** Edinburgh Math. Notes **37**, 7–9 (1949).

This is a simple proof using matrices of the Jacobi-Trudi identity

$$|\alpha^0\beta^r\gamma^2 \cdots \kappa^{n-1}|\,|h_0h_{r-1}h_{s-2} \cdots h_{t-n+1}| = |\alpha^0\beta^r\gamma^s \cdots \kappa^t|,$$

where $h_r$ is the $r$th complete homogeneous symmetric function in $n$ arguments $\alpha, \beta, \cdots, \kappa$. *C. C. MacDuffee.*

**Minagawa, Takizo. An elementary method to derive the normal form of $n$-dimensional real Euclidean rotation.** Kōdai Math. Sem. Rep., no. 2, 37–38 (1949).

The known expression of an $n$-dimensional rotation as a direct sum of plane rotations and perhaps a reflection is here derived by a method depending essentially on the principal axes of the quadratic form $x'Ax$, where $A$ is orthogonal. If the maximum of $x'Ax$ for $x'x=1$ is $\lambda$ and is achieved for $x=a$, then (i) $A=-1$ if $\lambda=-1$, (ii) $a$ is invariant under $A$ if $\lambda=+1$, and (iii) $A^2a-2\lambda Aa+a=0$ so that $a$ and $Aa$ span an invariant plane if $|\lambda|\neq 1$. *W. Givens.*

**Amante, S. Sul teorema di Amato relativo alle sostituzioni ortogonali periodiche di carattere assegnato.** Matematiche, Catania **3**, 1–9 (1948).

The author determines the number of nonequivalent orthogonal homographies in the space $S_{n-1}$ which are periodic of period $p$. For $p$ odd and greater than 1, this number is

$$\binom{(p-1)/2+[n/2]}{[n/2]} - \binom{(p-1)/2-\varphi(p)/2+[n/2]}{[n/2]}.$$

The formula is more complicated for $p$ even.

*C. C. MacDuffee* (Madison, Wis.).

**Amante, Salvatore. Sulla validità nel corpo reale di un teorema di Amato relativo alle sostituzioni ortogonali periodiche di carattere assegnato.** Matematiche, Catania **3**, 16–24 (1948).

V. Amato [Atti Accad. Gioenia Catania (5) **7**, no. 23 (1914)] showed that every complex orthogonal matrix of period $p$ can be written $S=A^{-1}\Phi A$, where $A$ is an arbitrary orthogonal matrix and $\Phi$ is uniquely defined by the multiplicities of the $p$th roots of unity which occur as characteristic roots of $S$. The present paper extends these results to the real field. *C. C. MacDuffee* (Madison, Wis.).

**Stewart, B. M. A note on least common left multiples.** Bull. Amer. Math. Soc. **55**, 587–591 (1949).

Let $A$ and $B$ be $n$-by-$n$ matrices with elements in a principal ideal ring and let

$$\begin{pmatrix} R & S \\ P & Q \end{pmatrix}\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where the first matrix on the left is unimodular and $D$ is in Hermite canonical triangular form; then if $A$ and $B$ are both nonsingular, $PA=-QB$ is a least common left multiple of $A$ and $B$. This usual restriction on $A$ and $B$ is here shown to be too strong, it being enough to require $D$ to be nonsingular. When $D$ is singular, $A$ and $B$ are replaced by their Hermite forms, $A_1$ and $B_1$, and then $PA_1=-QB_1$ is again the least common left multiple of $A$ and $B$ provided $(R\ S)$ is required to have a unit vector $e_i$ for its $i$th row whenever the $i$th row of $D$ is zero. *W. Givens.*

**Helsel, R. G. On the canonical reduction of quadratic forms.** Physical Rev. (2) **75**, 1088–1089 (1949).

An elementary proof of the known theorem: if $A$ and $B$ are Hermitian matrices of order $n$ and $A$ is positive definite, then the rank of $B-\lambda_i A$ is $n-k$, where $k$ is the multiplicity of the root $\lambda_i$ of $|B-\lambda A|=0$. *C. C. MacDuffee.*

**Leng, Sen-ming. A theorem of positive definite matrices.** Amer. Math. Monthly **56**, 397–398 (1949).

Let $A$ be a square matrix with determinant $|A|$. Set $|A|=|A|_+-|A|_-$, where $|A|_+$ denotes the sum of the terms which are given a positive sign in the determinant expansion. The author proves the following theorem. If $A$ is a positive definite Hermitian matrix, then $|A|_-\geqq 0$.

*G. B. Price* (Lawrence, Kan.).

**Mayer, Walther. The linear mappings of the $E_n$ into itself.** Monatsh. Math. **53**, 42–52 (1949).

Exposé de la réduction canonique d'une matrice carrée sur un corps qui se rapproche beaucoup de l'exposé classique de van der Waerden sauf qu'il n'est pas fait usage de la théorie des modules sur un anneau de polynomes. Le rapporteur remarque que l'exposé pourrait être abrégé en définissant dès le début le polynome caractéristique comme le polynome $P$ de plus petit degré tel que $P(A)=0$, puis en utilisant l'identité de Bezout pour décomposer l'espace en somme directe de sous-espaces invariants [théorème 8], ce qui est presque immédiat; on est ramené alors au cas où $P$ est puissance d'un polynome premier, cas traité dans le § 4.

*J. Dieudonné* (Nancy).

**Bell, James H. Left associates of monic matrices, with an application to unilateral matrix equations.** Amer. J. Math. **71**, 249–256 (1949).

A matrix $M$ is called monic of degree $k$ if it can be written as a polynomial in $\lambda$ of degree $k$ whose coefficients are $n$ by $n$ matrices with elements in a field $F$, the coefficient of $\lambda^k$ being the identity matrix. The problem considered is that of determining when a matrix $A$ in $F[\lambda]$ will be the left associate of a monic matrix of degree $k$. Assume that $A=(a_{rs})=A_s\lambda^s+\cdots+A_0$ is in (modified) Hermite form. If the degree of $a_{11}a_{22}\cdots a_{nn}$ is $kn$ and the degree of $a_{11}a_{22}\cdots a_{jj}$ is not greater than $kj$ for $j=1, \cdots, n-1$, then a necessary and sufficient condition is that the $kn$ by $sn$ matrix

$$\begin{Vmatrix} A_s & A_{s-1} & \cdots & A_{s-k+1} & \cdots & A_1 \\ 0 & A_s & \cdots & A_{s-k+2} & \cdots & A_2 \\ \cdots & \cdots & & \cdots & & \cdots \\ 0 & 0 & \cdots & A_s & \cdots & A_k \end{Vmatrix}$$

be of rank $kn$. *C. C. MacDuffee* (Madison, Wis.).

**Todd, J. A. A note on the algebra of $S$-functions.** Proc. Cambridge Philos. Soc. **45**, 328–334 (1949).

The $S$-functions are a variety of symmetric function whose importance lies in the fact that the $S$-function $\{\lambda\}$ of the latent roots of a matrix $A$ is equal to the spur of the invariant matrix $A^{(\lambda)}$ corresponding to the partition $(\lambda)$. Since an invariant matrix of an invariant matrix is also an invariant matrix and hence equivalent to a direct sum of irreducible invariant matrices, $[A^{(\lambda)}]^{(\mu)}$ is equivalent to

a direct sum of invariant matrices $A^{(\nu)}$, and one writes $\{\lambda\}\otimes\{\mu\}=\sum\{\nu\}$. The operation denoted by $\otimes$ was defined and discussed by the reviewer and is of considerable importance in invariant-theory since a knowledge of the expansion of $\{\lambda\}\otimes\{n\}$ implies a knowledge of all the concomitants of degree $n$ in a ground form of type $\{\lambda\}$, and for the general partition $(\mu)$, the expansion of $\{\lambda\}\otimes\{\mu\}$ gives a corresponding knowledge of concomitant types. Known methods of evaluating $\{\lambda\}\otimes\{\mu\}$, however, have not proved completely satisfactory as the orders become larger, and the author here suggests a new method. This consists in the evaluation of $\{\lambda\}\otimes S_n$, where $S_n$ is the power sum symmetric function. Thence, by making use of the fact that the operation $\otimes$ is distributive on the right with respect to both addition and multiplication, $\{\lambda\}\otimes\{\mu\}$ can be evaluated from the expression for $\{\mu\}$ in terms of the $S_r$'s. The author shows that the expression for $\{\lambda\}\otimes S_n$ in terms of the $S_r$'s can be obtained from that for $\{\lambda\}$ by replacing each $S_r$ by $S_{rn}$. Use is then made of a formula obtained by the reviewer and Richardson for the characters of a class of the symmetric group for which the order of every cycle is divisible by $n$. Using the orthogonal properties of the symmetric group characters an expansion for $\{\lambda\}\otimes S_n$ is obtained. Recurrence relations are also obtained to give $\{m\}\otimes S_n$ from $\{m-1\}\otimes S_n$ and $\{m\}\otimes S_{n-1}$. This provides a reasonable, though not too easy, method of systematically computing the expansions $\{m\}\otimes\{\lambda\}$. *D. E. Littlewood.*

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### *Abstract Algebra*

***Birkhoff, Garrett. Lattice Theory.** American Mathematical Society Colloquium Publications, vol. 25, revised edition. American Mathematical Society, New York, N. Y., 1948. xiii+283 pp. $6.00.

This second edition of Lattice Theory is more than just a revision of the first edition of 1940 [these Rev. 1, 325]. It has doubled in size. Eight of the sixteen chapters are new, and the other chapters have all been rewritten and expanded. This increase in size reflects the rapid growth of the subject during the intervening years. Titles of the new chapters are: Chains and chain conditions, Complete lattices, Applications to algebra, Semi-modular lattices, Applications to set theory, Lattice-ordered semigroups, Lattice-ordered groups, and Ergodic theory. A prominent new feature consists of hundreds of exercises distributed throughout the book, many of them containing new results both of the author and of others. Along with the exercises 111 unsolved problems are proposed, replacing the 17 problems of the original edition, 8 of which have since been solved.

There are two forewords, on algebra and topology. The first chapter, on partially ordered sets (or partly ordered sets, as they are called here) now treats the extended cardinal and ordinal arithmetic due to the author and studied also by M. M. Day. The second chapter on lattices includes results of the author on neutral elements and of Whitman on free lattices. Chapter III on chain conditions has a new treatment of Zorn's lemma, well-ordering, and the axiom of choice. The fourth chapter on complete lattices contains new material on polarities and Galois connections, as well as on representations and intrinsic topologies of lattices and fixed point theorems.

The sixth chapter on applications to algebra is new. It deals with quasi-groups and loops, new extensions of the Jordan-Hölder and Kurosh-Ore theorems, and the classification of groups by their subgroup lattices. A new formulation, due to the author, of the theorem of Ore is proved: in a modular lattice of finite length, the elements of any two representations of the unit element as a direct join of indecomposable elements are projective in pairs. Three theorems on algebras with commutative congruence relations, announced at the Princeton Bicentennial Conference in 1946, are published here for the first time. The author's theorem on subdirect unions of subdirectly irreducible algebras is also proved.

Chapter VII on semi-modular lattices (sometimes called Birkhoff lattices) discusses extensions of the notion of semi-modularity to lattices of infinite length, due to Wilcox and the author. It also deals with the lattice theory of linear dependence, lattices of subspaces of projective and affine spaces, and a theory of non-Desarguesian plane geometries. The theorems of Whitman, that every lattice is isomorphic to a sublattice of a lattice of partitions, and of Dilworth, that every finite lattice is isomorphic to a sublattice of a semi-modular lattice, are considered.

The eighth chapter on complemented modular lattices contains material on projective spaces of infinite dimension due to Prenowitz and the reviewer. New material in the tenth chapter on Boolean algebras deals with postulate systems, Boolean equations, and Boolean $\sigma$-algebras.

The eleventh chapter on applications to set theory is new. It takes up Stone's application of Boolean algebra to topology, Kaplansky's theory of the lattices of real-valued continuous functions over compact Hausdorff spaces, the abstract closure algebras of McKinsey and Tarski, and a new treatment of measure theory. Chapter XIII on lattice-ordered semigroups develops the theory of residuated lattices studied by M. Ward and Dilworth. It also deals with the algebra of relations and Boolean matrices.

Much of the material in the new fourteenth chapter on lattice-ordered groups is due originally to the author [Ann. of Math. (2) 43, 298–331 (1942); these Rev. 4, 3]. This theory is now integrated with the work of others, and some new results and extensions of old results appear for the first time. The following generalization of a theorem of Everett and Ulam on ordered groups is proved: any subgroup of a $\sigma$-complete partially ordered group is Archimedean and integrally closed, and conversely the completion by nonvoid cuts of any integrally closed partially ordered group is a complete partially ordered group.

Chapter XV on vector lattices reflects the great amount of work that has appeared on partially ordered linear spaces since 1940, for example, by Kakutani, Bohnenblust, and M. and S. Kreĭn. The treatment of ergodic theory in the last chapter of the book has also undergone revision in the light of more recent work.

The bibliography has been enlarged, but still contains only the more important references. Hundreds of other references are given in footnotes. The subject index and author index have also been greatly expanded. The book as a whole is a useful, complete and up-to-date account of a new and active branch of abstract algebra.

*O. Frink* (State College, Pa.).

**Piaget, Jean. Le groupe des transformations de la logique des propositions bivalentes.** Arch. Sci. Soc. Phys. Hist. Nat. Genève 2, 179–182 (1949).

The author defines three transformations called the inverse, reciprocal, and correlative, which are applicable to

the sixteen Boolean functions of two variables. Together with the identity, they form an Abelian group (the four-group). He notes a similarity between these transformations and others which are definable for the systems called "groupements" which he has previously considered.

*O. Frink* (State College, Pa.).

**Livšic, A. H. On the Jordan-Hölder theorem in structures.** Mat. Sbornik N.S. **24(66)**, 227–235 (1949). (Russian)

In contrast to all forms of the Jordan-Hölder Theorem for lattices which use a definite relation of normality, A. I. Uzkov [Rec. Math. [Mat. Sbornik] N.S. **4(46)**, 31–43 (1938)] considered lattices with an undefined relation $aNb$ ("$b$ normal in $a$") and set up necessary and sufficient conditions on this relation of normality for the J.-H. theorem, more precisely the Schreier refinement theorem and Zassenhaus construction, to be valid. However, he imposed the preliminary restriction (A) that to any two elements $a$ and $b$ there are elements in the lattice normal both in $a$ and in $b$. Later Kořínek [Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1941**, no. 14 (1942); these Rev. **7**, 509] generalized Uzkov's theorem, simplified its proof, and replaced the restriction (A) by the somewhat weaker one (B) that to any two elements $a$ and $b$ there are elements in the lattice normal in $a$ and preceding $b$ (in the lattice order).

The present author shows by an example that even this weaker condition (B) is not required for the J.-H. theorem. He then develops a set of necessary and sufficient conditions on the relation of normality free from any preliminary restriction. If condition (B) holds in a lattice then his main theorem becomes equivalent to Kořínek's theorem. Notation: the order-relation in the lattice $S$ is denoted by $\leqq$; the lattice operations by $+$ and $\cdot$; the set of elements $x$ of $S$ with $b \leqq x \leqq a$ is the segment $a/b$. The undefined relation of "normality" holds between some pairs $a$, $b$ of $S$ with $a \geqq b$; $aNb$ need not be transitive. Definitions: $a=a_0 \leqq a_1 \leqq \cdots \leqq a_n=b$ with $a_{i+1}Na_i$ is a "normal series" between $a$ and $b$ with the "factors" $a_{i+1}/a_i$. The segment $a/b$ is an extension of $c/d$ if $a=b+c$, $d=bc$, and the mapping $x \to y=x+b$ establishes a one-to-one relation whose inverse mapping is given by $y \to x=cy$ so that $x=c(x+b)$. Two segments $a/b$ and $c/d$ are "isomorphic" extensions of a third $u/v$, if the one-to-one relation between their elements which is set up naturally by means of the corresponding elements in $u/v$ is preserved under lattice operations and if normal pairs of the one correspond to normal pairs of the other. From two normal series of $S$ (1) $a_0 \leqq a_1 \leqq \cdots \leqq a_r$ and (2) $b_0 \leqq b_1 \leqq \cdots \leqq b_s$ two systems of "Zassenhaus series" (3) $a_{i0} \leqq a_{i1} \leqq \cdots \leqq a_{is}$ $(i=0, 1, \cdots, r-1)$ and

$$(4) \qquad b_{j0} \leqq b_{j1} \leqq \cdots \leqq b_{jr}$$

$(j=0, 1, \cdots, s-1)$ are obtained, where $a_{ij}=a_i+a_{i+1}b_j$ and $b_{ji}=b_j+b_{j+1}a_i$. The element $a$ is "$\beta$-Dedekindian" with respect to $b$ and $c$, if $a \leqq b$ and $b(a+c)=a+bc$; and it is "$\delta$-Dedekindian" if $a \geqq b$ and $a(b+c)=b+ac$. [Compare in this connection O. Ore, Trans. Amer. Math. Soc. **41**, 266–275 (1937).] Theorem. The Zassenhaus series (3) and (4) of any two normal series (1) and (2) are normal, the pairs of corresponding factors $a_{i,j+1}/a_{ij}$ and $b_{j,i+1}/b_{ji}$ are isomorphic, and the factor whose extension they both are has the form $a_{i+1}b_{j+1}/(a_ib_{j+1}+a_{i+1}b_j)$, if and only if the relation of normality satisfies the following conditions for any four elements $x_1, x_2, y_1, y_2$ of $S$: (i) $x_1Nx_2$ and $x_1 \geqq y_1Ny_2$ implies $(x_2+y_1)N(x_2+y_2)$; (ii) $x_1Nx_2$ and $y_1Ny_2$ implies $x_1y_1N(x_1y_2+x_2y_1)$; (iii) $x_1Nx_2$ and $y_1Ny_2$ implies that every element $y$ with $x_1y_1 \geqq y \geqq x_1y_2+x_2y_1$ is $\beta$-Dedekindian with respect to $x_1y_1$ and $x_2$; (iv) $x_1Nx_2$ and $y_1Ny_2$ implies that every element $x$ with $x_2+x_1y_1 \geqq x \geqq x_2+x_1y_2$ is $\delta$-Dedekindian with respect to $x_2$ and $x_1y_1$.

In the last part of the paper some results of A. Kuroš on infinite normal and composition systems in groups [Rec. Math. [Mat. Sbornik] N.S. **16(58)**, 59–72 (1945); these Rev. **8**, 253] are transferred to lattice theory.

*K. A. Hirsch* (Newcastle-upon-Tyne).

**Stoll, R. R. Equivalence relations in algebraic systems.** Amer. Math. Monthly **56**, 372–377 (1949).

Expository article.

**Gonzalez Baz, Enriqueta. Considerations on distributive systems.** Comisión Impulsora y Coordinadora de la Investigación Científica (Mexico). Anuario **1945**, 31–42 (1947). (Spanish)

The author considers systems which are closed under $+$ and $\cdot$, each of these operations being left and right distributive with respect to the other, with elements 0 and $I$ such that $x+0=x=0+x$ and $x \cdot I=x=I \cdot x$ for all $x$ in the system (but associativity and commutativity are not postulated, nor are $x \cdot 0=0$, $x+I=I$). The free system of this type generated by $0, I, a_1, \cdots, a_{n-2}$ consists of the

$$\sum_{\mu=1}^{2^n} \prod_{\nu=1}^{2^n} x_{\mu,\nu} \equiv \sum_{\mu=1}^{2^{n-1}} \prod_{\nu=1}^{2^n} x_{\mu,\nu} + \sum_{\mu=1}^{2^{n-1}} \prod_{\nu=1}^{2^{n2^{n-1}+\mu}} x_{2^{n-1}+\mu,\nu};$$

$$\sum_{\mu=1}^{2^0} \prod_{\nu=1}^{2^{n_0}} x_{\mu,\nu} \equiv \prod_{\nu=1}^{2^{n_1}} x_{1,\nu} \equiv \prod_{\nu=1}^{2^{n_1-1}} x_{1,\nu} \cdot \prod_{\nu=1}^{2^{n_1-1}} x_{1,2^{n-1}+\nu},$$

where the $x_{\mu,\nu}$ are generators. However, not all these forms represent distinct elements: $I \cdot I \cdot I \cdot I = I \cdot I$. The system generated by three elements $0, I, a$ is determined explicitly, consisting of the seven elements $0, I, a, 0 \cdot a, I+a, 0 \cdot a \cdot 0, I+a+I$.

*P. M. Whitman* (Silver Spring, Md.).

**Heller, Isidor. On generalized polynomials.** Rep. Math. Colloquium (2) **8**, 58–60 (1948).

In a tri-operational algebra of K. Menger [same Rep. (2) **5–6**, 3–10 (1944); **7**, 46–60 (1946); these Rev. **6**, 143; **8**, 61] an element $p$ is called a generalized polynomial if for every element $f$ there exist constants $c_0, \cdots, c_n$ such that $pf=c_0+c_1f+\cdots+c_nf^n$; while $p$ is called a polynomial if the constants $c_0, \cdots, c_n$ are the same for all $f$. In the present note the author exhibits a tri-operational algebra in which every nonconstant element is a generalized polynomial without being a polynomial. He also shows that, if the algebra contains a universal element $f$ such that for every element $g$ and constant $c$ there is a constant $d$ so that $gc=fd$, then every generalized polynomial is a polynomial.

*O. Frink* (State College, Pa.).

**Snapper, Ernst. Completely indecomposable modules.** Canadian J. Math. **1**, 125–152 (1949).

Let $A$ be a commutative ring with unit element. An Abelian group $V$ which has $A$ as operator domain (with unit element as unit operator) is said to be a completely indecomposable module if the invariant submodules of $B$ satisfy both chain conditions and if every invariant submodule of $V$ is indecomposable. Two such $A$-modules are operator isomorphic if and only if they have the same annihilating ideal. A direct sum of completely indecomposable modules is said to be regular. The theory of regular modules contains as a special case the classical theory of a

linear transformation on a finite dimensional vector space. In this special case every completely indecomposable module is cyclic. Examples are given of more general rings for which not every indecomposable module is cyclic. In general a completely indecomposable module is cyclic if and only if its annihilating ideal is intersection irreducible. A concept of interlacing is introduced for modules over an arbitrary ring $B$. Let $V_1$ and $V_2$ be $B$-modules and let $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ be invariant submodules isomorphic under a mapping $J$. In the direct sum $V_1+V_2$ the set of elements $(w_1, -w_1J)$, $w_1 \epsilon W_1$, form an invariant submodule $W$. The residue class module $V=(V_1+V_2)-W$ is said to have been obtained by interlacing $V_1$ and $V_2$ under the lacing isomorphism $J$. The operation of lacing is applied to prove that every commutative completely primary ring $A$ with unit element, whose ideals satisfy both chain conditions, has a completely indecomposable module as a faithful representation space. The module thus constructed provides in a sense a generalization to modules over a ring of the classical construction of dual spaces for modules over an algebra.

*R. M. Thrall* (Ann Arbor, Mich.).

**Chatland, Harold, and Mann, H. B. Integral extensions of a ring.** Bull. Amer. Math. Soc. **55**, 592–594 (1949).

Let $R$ be a commutative ring with unit element. An overring $R'$ of $R$ is called an integral extension of $R$ if whenever $a/b$ is in $R'$ (where $a, b \epsilon R$) then $a/b$ is in $R$. It is proved that (I) necessary and sufficient that two elements $a$ and $b$ of $R$ have no common divisor in any integral extension of $R$ is that there exist $x, y$ in $R$ such that $ax+by=1$, and (II) to every set of ideals of $R$ there exists an integral extension $R'$ such that every ideal of the set is the intersection of $R$ with a principal ideal of $R'$. *I. S. Cohen.*

**Fuchs, Ladislas. The extension of the notion "relatively prime."** Acta Univ. Szeged. Sect. Sci. Math. **13**, 43–47 (1949).

The author generalizes to non-Noetherian rings the results of a previous paper [Norske Vid. Selsk. Forh., Trondhjem **20**, no. 7, 25–28 (1947); these Rev. **10**, 6]. If $A$ is an ideal in a ring $R$, then $b$ is said to be primary to $A$ if $A:b$ is contained in the radical of $A$. Then $b$ is primary to $A$ if and only if it lies in no isolated primary component of $A$. The element $b$ is said to be almost prime to $A$ if $b$ is prime to the radical of $A$; this will be true if and only if every power of $b$ is primary to $A$. The properties "prime to $A$" and "almost prime to $A$" coincide if $A$ is the intersection of its isolated primary components. Using the above concepts, and placing some restrictions on $R$, the author gives some characterizations of primary and quasiprimary ideals. *I. S. Cohen* (Cambridge, Mass.).

**Nakayama, Tadasi. Commuter systems in a ring with radical.** Duke Math. J. **16**, 331–337 (1949).

Let $R$ be a ring with unit element 1 and minimal condition, and $K$ any subring containing 1. Let $A$ be the absolute endomorphism ring of the additive group of $R$, taking elements of $A$ as right operators. Let $R_r$ (respectively $K_r$) denote the subring (in $A$) of right multiplications by elements of $R$ (or $K$), and $R_l$ and $K_l$ the rings of left multiplications. Call an $R$-right-module $R$-regular when, for some $m$ and $n$, the direct sum of $m$ copies of it is $R$-isomorphic to the direct sum of $n$ copies of $R$. Let $V_X(Y)$ denote the commuter [commutator] of $Y$ in $X$. The author proves that if (1) $R$ is $R_rK_l$-right regular, and (2) every ring between $R_rK_l$ and $R_r$ for which $R$ is regular is of form $R_rL_l$ for an $L \subseteq K$ with $R_rL_l \cap K_l = L_l$, then there is a one-to-one correspondence between subrings $L$ such that $R_rL_l \cap K_l = L_l$ and $R$ is $R_rL_l$-regular, and subrings $T \supseteq V_R(K)$ for which $R$ is $T$-left regular; the correspondence is given by $V_K(T)=L$, $V_R(L)=T$. If $K$ contains the center of $R$ and $R_rK_l$ has an independent finite basis over $R_r$ consisting of elements of $K_l$ (in particular, if $R$ is simple) and if (1) is satisfied, then so is (2). A further generalization, weakening (1), is mentioned.

Previous theorems on dualism of subrings with their commutators (Artin and Whaples [Amer. J. Math. **65**, 87–107 (1943); these Rev. **4**, 129], Azumaya [quoted by the author but apparently unpublished], Asano [Jap. J. Math. **15**, 231–253 (1939); these Rev. **1**, 103] and others) assumed some or all of $R$, $T$, $L$ to be simple or to be of finite rank, and are consequences of the theorem proved here. *G. Whaples* (Bloomington, Ind.).

**Hua, Loo-Keng. On the automorphisms of a sfield.** Proc. Nat. Acad. Sci. U. S. A. **35**, 386–389 (1949).

A semi-automorphism of a division ring is an additive automorphism preserving the unit element and the combination $aba$. The author gives a brief proof that such a mapping is either an automorphism or an anti-automorphism. For algebras of finite order, this had been proved by Ancochea [Ann. of Math. (2) **48**, 147–153 (1947); these Rev. **8**, 310] and the reviewer [Duke Math. J. **14**, 521–525 (1947); these Rev. **9**, 172]. Extension of the theorem to semi-simple rings with chain condition is indicated.

*I. Kaplansky* (Chicago, Ill.).

**Cohn, Richard M. A theorem on difference polynomials.** Bull. Amer. Math. Soc. **55**, 595–597 (1949).

It is proved that every system of algebraic ordinary difference equations in $y_1, \cdots, y_n$ is equivalent to such a system consisting of $n+1$ equations, provided the difference field of coefficients contains an element distinct from its transforms of all orders. Although the analogous theorem was known for algebraic equations [Kronecker] and for algebraic differential equations [Ritt], the result is somewhat surprising for the case of difference equations because it is known that a perfect difference ideal of difference polynomials in $y_1, \cdots, y_n$ can have a basic set of arbitrary length. *E. R. Kolchin* (New York, N. Y.).

**Cohn, Richard M. Inversive difference fields.** Bull. Amer. Math. Soc. **55**, 597–603 (1949).

Let $F$ be an ordinary difference field, let $A$ be an irreducible difference polynomial in $y_1, \cdots, y_n$ with coefficients in $F$, and for each $y_i$ effectively present in $A$ let $u_i$ and $v_i$ be the transforms of $y_i$ of lowest and highest orders, respectively. The author has previously shown [same Bull. **54**, 917–922 (1948); these Rev. **10**, 178] that in the prime decomposition of the perfect difference ideal $\{A\}$ each of the prime difference ideals contains either none or all of the difference polynomials $\partial A/\partial v_1, \cdots, \partial A/\partial v_n$, and that the solutions of one of these prime ideals constitute an ordinary or a singular manifold of $A$ depending on whether the former or latter possibility occurs. He has also shown that the number of ordinary manifolds is greater than or equal to 1 and not greater than the degree of $A$ in $v_i$. In the present paper it is shown that this number is not greater than the degree of $A$ in $u_i$. This is done by observing that $F$ can always be imbedded in a difference field in which every element is a transform (an inversive difference field),

and then setting up a sort of duality between the transforming operation and its inverse. *E. R. Kolchin.*

**Hochschild, G. Double vector spaces over division rings.** Amer. J. Math. 71, 443–460 (1949).

Soit $D$ un corps (commutatif ou non), $E$ un anneau ayant un élément unité. L'auteur considère les bimodules $M$ sur $D$ (à gauche) et $E$ (à droite), et spécialement ceux qui sont engendrés par un seul élément $m_0$ (c'est-à-dire dont tout élément est de la forme $\sum_i \alpha_i m_0 \beta_i$ avec $\alpha_i \in D$ et $\beta_i \in E$); ces derniers sont dits cycliques et généralisent les "composites" de N. Jacobson [same J. 66, 1–29 (1944); ces Rev. 6, 35]. À un tel bimodule cyclique l'auteur associe son "espace des relations" $R(M, m_0)$: si $M'$ est le dual de $M$ (ce dernier considéré comme espace vectoriel à gauche sur $D$), à tout $x' \in M'$ l'auteur associe l'homomorphisme $\sigma$ du groupe additif $E$ dans le groupe additif $D$, défini par $\sigma(u) = x'(m_0 \cdot u)$ pour tout $u \in E$; il obtient ainsi une correspondance biunivoque entre bimodules cycliques et certains sous-groupes du groupe des homomorphismes de $E$ (groupe additif) dans $D$ (groupe additif) qu'il caractérise lorsque $M$ est de dimension finie sur $D$. Ces notions lui permettent de généraliser la théorie correspondante développée par Jacobson pour le cas où $D = E$ est un corps commutatif [loc. cit.]. L'auteur étudie aussi d'autres types de bimodules qu'il nomme réguliers: $E$ est alors un sous-corps de $D$, et on suppose qu'il existe un sous-corps $K$ de $E$ et un système de générateurs $V_0$ du bimodule $V$ considéré, tel que $kv = vk$ pour $k \in K$ et $v \in V_0$. Lorsque $D$ est galoisien et de rang fini sur $K$ (au sens de la théorie de Cartan [Ann. Sci. École Norm. Sup. (3) 64, 59–77 (1947); ces Rev. 9, 325] et Jacobson [même J. 69, 27–36 (1947); ces Rev. 9, 4]), on peut décrire complètement la structure des bimodules réguliers; l'auteur déduit d'ailleurs de cette étude un certain nombre de résultats de Cartan-Jacobson. Signalons aussi qu'il donne une démonstration élémentaire du "théorème de Jacobson-Bourbaki" qui est à la base de cette théorie (la démonstration de Jacobson repose sur la théorie des anneaux simples, et celle de N. Bourbaki sur la théorie des éléments primordiaux).

*J. Dieudonné* (Nancy).

**Papy, Georges. Sur les formes extérieures définies sur un anneau d'intégrité à idéaux principaux.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) 35, 157–176 (1949).

The author develops the theory of Grassmann algebras over a domain of integrity in which every ideal is principal, and shows that many classical theorems concerning matrices can be derived in an elegant way as special cases of theorems on exterior forms. *E. R. Kolchin* (New York, N. Y.).

**Brauer, Richard. On a theorem of H. Cartan.** Bull. Amer. Math. Soc. 55, 619–620 (1949).

L'auteur donne une démonstration élémentaire (indépendante de la théorie de Galois) du théorème suivant. Si $K$ est un corps (skew field) de centre $C$, et $H$ un sous-corps stable par tous les automorphismes intérieurs de $K$, alors $H = K$ ou $H \subset C$. L'hypothèse (faite par H. Cartan) que $K$ est de rang fini sur $C$ n'intervient pas dans la démonstration, qui est basée sur une simple identité algébrique, et vaut, plus généralement, lorsque $K$ et $H$ sont des anneaux satisfaisant à certaines conditions simples. *H. Cartan.*

**Kuročkin, V. M. Rings with the minimal condition for quasi-ideals.** Doklady Akad. Nauk SSSR (N.S.) 66, 549–551 (1949). (Russian)

The author is pursuing Andrunakievič's program [Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 129–178 (1948); these Rev. 9, 564] of studying rings relative to the operation $aob = a + b - ab$. A right quasi-ideal is a subset closed under $a - b + c$ and containing $aox$ along with $a$. Theorem 1. If $A$ is semi-simple in the sense of Jacobson and satisfies the minimal condition for right quasi-ideals, then $A$ satisfies the minimal condition for ordinary right ideals. The proof follows the lines of Jacobson's technique, the primitive case being first treated by using a representation as a dense ring of linear transformations. Theorem 2. If $A$ satisfies the minimal condition for right quasi-ideals, and has no two-sided quasi-ideals other than $A$, then $A$ is a radical ring. This is an immediate consequence of theorem 1.

*I. Kaplansky* (Chicago, Ill.).

**Wever, Franz. Über Invarianten in Lie'schen Ringen.** Math. Ann. 120, 563–580 (1949).

If $L$ is a free Lie ring with $k$ free generators $\xi_1, \cdots, \xi_k$, and $W_m(\xi_1, \cdots, \xi_k)$ is a homogeneous polynomial of dimension (degree) $m$ with ordinary integral coefficients, then a unimodular linear transformation $\xi_i = \sum a_{ij}\eta_j$ replaces $W_m(\xi_1, \cdots, \xi_k)$ by $V_m(\eta_1, \cdots, \eta_k)$. We call $W_m$ an invariant of $L$ if $V_m(\xi_1, \cdots, \xi_k) = \pm W_m(\xi_1, \cdots, \xi_k)$. The ring $L$ has a faithful imbedding $\alpha \to a$ in the free associative ring $R$ of all polynomials in $k$ free generators $x_1, \cdots, x_k$. The mapping is defined by $\xi_i \to x_i$, $\alpha + \beta \to a + b$, $\alpha\beta \to ab - ba$. The author uses this mapping and the result of Schur on the characters of the rational representation of the general linear group to compute the number of invariants of dimension $m$ in $L$. For $k > 2$ the results are expressed in terms of the number of one-dimensional representations. Explicit numerical formulas are given for $k = 2$. *A. A. Albert* (Chicago, Ill.).

**Barsotti, Iacopo. Algebre senza base finita. II.** Ann. Mat. Pura Appl. (4) 27, 243–260 (1948).

[Part I appeared in the same Ann. (4) 26, 57–66 (1947); these Rev. 10, 98.] Continuing his discussion of algebras of infinite order, the author takes up ideals, subalgebras, idempotents and Kronecker products; the latter are constructed without using a basis. *I. Kaplansky* (Chicago, Ill.).

**Schafer, R. D. The Wedderburn principal theorem for alternative algebras.** Bull. Amer. Math. Soc. 55, 604–614 (1949).

If $\mathfrak{A}$ is an alternative algebra with radical $\mathfrak{N}$ such that $\mathfrak{A} - \mathfrak{N}$ is separable, then the author proves that there exists an algebra $\mathfrak{S}$ equivalent to $\mathfrak{A} - \mathfrak{N}$ such that $\mathfrak{A} = \mathfrak{S} + \mathfrak{N}$. This is a generalization of a theorem of Wedderburn for associative algebras. The proof is similar to the associative one. *R. E. Johnson* (Northampton, Mass.).

**Raffin, Raymond. L'inversibilité dans les algèbres linéaires non associatives.** C. R. Acad. Sci. Paris 228, 1685–1687 (1949).

The author defines three classes of nonassociative algebras, each of which coincides in the associative case with the class of all (associative) division algebras. The principal results concern algebras of degree two over a real field.

*R. D. Schafer* (Philadelphia, Pa.).

**Raffin, Raymond. Algèbres symétriques à division d'ordre quatre.** C. R. Acad. Sci. Paris 228, 1788–1790 (1949).

Continuing the investigation begun in the preceding note, the author obtains a result for algebras of degree two and order four which is essentially the same as a theorem of Dickson [Duke Math. J. 1, 113–125 (1935), theorem 2].

*R. D. Schafer*

**Etherington, I. M. H. Non-associative arithmetics.** Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 442–453 (1949).

The author considers a mathematical system $A$ consisting of a set of elements and the binary operations of addition and multiplication. He also assumes that there is an equivalence relation called equality which is consistent with the two operations. Then $A$ is said to satisfy the full right distributive law if it is true that for every $a$ and sum $\sum b_i$ of $A$ we have $a(\sum b_i)=\sum ab_i$, where the order and grouping is the same in the two sums. The system $A$ is now called an arithmetic if the full right distributive law holds, $A$ has a unity element 1 such that $1\cdot a=a\cdot 1=a$ for every $a$ of $A$, and every element of $A$ is a finite sum with all summands equal to 1. Every arithmetic satisfies the associative law for multiplication. The system of partitioned serials is the most general system in the sense that every arithmetic is a homomorphic image of a subsystem of it. The author concludes the paper by a study of the homomorphism relations between a number of special arithmetics defined by additional postulates. *A. A. Albert* (Chicago, Ill.).

## THEORY OF GROUPS

**Kuroš, A. G., and Černikov, S. N. Solvable and nilpotent groups.** Uspehi Matem. Nauk (N.S.) **2**, no. 3(19), 18–59 (1947). (Russian)

The discovery and study of meaningful generalizations of the types of groups mentioned in the title have led the authors to this broad survey of concepts, methods and problems in contemporary group theory which extend the ideas of commutativity and finiteness. The paper is said to be developed from lectures delivered by one of the authors, in 1945–1946, at Moscow University, but based on their joint research. Among the references are Kuroš [Rec. Math. [Mat. Sbornik] N.S. **5(47)**, 347–354 (1939); these Rev. **2**, 2], Černikov [ibid. **6(48)**, 199–214 (1939); **7(49)**, 35–64 (1940); these Rev. **1**, 162; **2**, 5], and Baer [Trans. Amer. Math. Soc. **47**, 393–434 (1940); Bull. Amer. Math. Soc. **50**, 143–160 (1944); these Rev. **2**, 1; **5**, 227]. An attractive feature is the clear presentation of over thirty unsolved problems. Three schematic diagrams aid the reader in relating known results (of which several are new), to many of the plausible conjectures which form the content of the unsolved problems. A brief survey of the subject matter of each of the eleven sections of the paper follows.

Section 1 is concerned with unsolved problems about finiteness conditions on groups. Burnside's problem is one example. Another asks whether a group is finite if each ascending and each descending chain of subgroups is finite. Sections 2 and 3 treat generalizations of the Jordan-Hölder-Schreier theorem. Normal series are replaced by normal systems, defined as sets of subgroups, partially ordered by inclusion, complete, containing the two nonproper subgroups and with the property [Baer, first reference above, p. 395] that covering by inclusion implies normal inclusion. Generalizations of principal series, invariant series and composition series to like systems are also studied. Each of these leads, in section 4, to an extension of the concept of solvability, with the inevitable questions of equivalence and interrelation of these concepts. Subgroups, factor groups and direct sums of the above groups are studied in section 5, where it is proved that the direct sum of any set of solvable groups (solvability based on composition systems) is so solvable. Defining a local system to be a set of subgroups which cover the group and are directed by inclusion, the authors prove, in section 6, that if a group has a local system, each member of which is solvable (solvability based on normal systems), then the group itself is so solvable. In section 7, finiteness and solvability notions are combined. A solvable group (of the type discussed just above), in which every descending chain of subgroups is finite, is solvable in the classical sense. In section 8, groups are studied every subgroup of which lies in a normal system. In section 9, ascending central series are generalized to so-called central systems, thus extending the notion of nilpotent group. A typical unsolved problem is the following. Let $G$ be a group in which each invariant system can be refined to a central system. Do all its subgroups have this property? Solvability and nilpotency are considered together in section 10. If a group has a central system which is an increasing invariant series, then it contains such a series with cyclic factors. Since nilpotent and special groups coincide in the finite case [Zassenhaus, Lehrbuch der Gruppentheorie, v. 1, Teubner, Leipzig-Berlin, 1937, p. 107], certain direct sums are treated in section 11. *F. Haimo* (St. Louis, Mo.).

**Čarin, V. S. A remark on the minimal condition for subgroups.** Doklady Akad. Nauk SSSR (N.S.) **66**, 575–576 (1949). (Russian)

An example is constructed to illustrate the existence of a solvable periodic group which satisfies the minimal condition for normal subgroups, but does not satisfy the minimal condition for subgroups. *R. A. Good.*

**Čarin, V. S. On complete groups with a radical series of finite length.** Doklady Akad. Nauk SSSR (N.S.) **66**, 809–811 (1949). (Russian)

A group $\mathfrak{G}$ has a "radical series" of length $n$ if it possesses a sequence of subgroups $1=\mathfrak{G}_0\subset\mathfrak{G}_1\subset\cdots\subset\mathfrak{G}_{n-1}\subset\mathfrak{G}_n=\mathfrak{G}$ where each term is normal in the following and the factor-groups $\mathfrak{G}_{i+1}/\mathfrak{G}_i$ are isomorphic either to the additive group of rational numbers or to a group of type $p^\infty$ for some prime number $p$. The following results are announced. If some set $\mathfrak{A}$ of quadratic nonsingular matrices over the field of rational numbers constitutes a complete group with an ascending central series, then all eigenvalues of each matrix $A$ of $\mathfrak{A}$ are equal to 1. If a complete group of nonsingular matrices over the field of rational numbers possesses an ascending central series, then it is equivalent to a group of triangular matrices with units in the main diagonal and rational coefficients. If a group $\mathfrak{G}$ possesses a radical series of finite length, then it possesses an ascending central series of finite length. The proof of this last main theorem is based chiefly on S. N. Černikov's theory of complete groups [Mat. Sbornik N.S. **22(64)**, 319–348 (1948); these Rev. **9**, 566].

*K. A. Hirsch* (Newcastle-upon-Tyne).

**Prokof'ev, A. N. On the fundamental theorem of Frobenius.** Doklady Akad. Nauk SSSR (N.S.) **65**, 801–804 (1949). (Russian)

The theorem of Frobenius is concerned with the number of solutions in a group of equations of the form $x^n=a$. Numerous earlier authors [Weisner, Turkin, Dubuque, Tchernikoff, Grosheff] have given successively sharper results. In a different direction, P. Hall [Proc. London Math. Soc. (2) **40**, 468–480 (1935); 481–501 (1936)] generalized by replacing $x^n=a$ by an arbitrary equation, and

sorted out the solutions in double cosets. The author announces four theorems extending these results, and in part uniting them. No proofs are given. *I. Kaplansky.*

**Čunihin, S. A. On theorems of Sylow's type.** Doklady Akad. Nauk SSSR (N.S.) **66**, 165–168 (1949). (Russian)

Let the group $\mathfrak{G}$ have order $g$. Let $\Pi$ be an arbitrary nonvoid set of prime numbers. Let $m$ be the maximal divisor of $g$ such that $(m, g/m)=1$ and such that either $m=1$ or else every prime factor of $m$ belongs to $\Pi$. If $\mathfrak{G}$ contains at least one solvable subgroup of order $m$ and if all subgroups of order $m$ are conjugate in $\mathfrak{G}$, then $\mathfrak{G}$ is said to be of type 1 relative to $\Pi$. The first theorem states that if $\mathfrak{G}$ has a normal series every factor of which is of type 1 relative to $\Pi$, then $\mathfrak{G}$ is of type 1 relative to $\Pi$. The second theorem asserts an analogous result for groups of type 1 relative to every nonvoid subset $\Pi_1 \subseteq \Pi$. *R. A. Good* (College Park, Md.).

**Jordan, Pascual. Zur Begründung der Darstellungstheorie endlicher Gruppen.** Z. Naturforschung **3a**, 522–523 (1948).

This note contains a short direct proof of the classical theorem that a unitary group of $n$-rowed matrices is irreducible over the complex numbers only if it contains $n^2$ linearly independent matrices. *R. M. Thrall.*

**Robinson, G. de B. On the representations of the symmetric group. III.** Amer. J. Math. **70**, 277–294 (1948).

This continues the study begun in the second paper [same J. **69**, 286–298 (1947); these Rev. **8**, 563] of representations of the symmetric group associated with skew diagrams. Especial attention is given to the case in which the disjoint constituents of a skew diagram are right diagrams. This case includes the theory of decompositions of Kronecker products of irreducible representations of the full linear group. Let $(\alpha)$ denote a partition of $n$ and $\chi_{(\alpha)}$ the corresponding irreducible character of $S_n$. Then the multiplicity of $\chi_{(\gamma)}$ in the skew representation $\chi^{(\beta)}_{(\alpha)}$ is equal to the multiplicity of $\chi_{(\alpha)}$ in $\chi_{(\beta)}\chi_{(\gamma)}$. Nakayama's idea of hook [Jap. J. Math. **17**, 165–184 (1940); these Rev. **3**, 195; **4**, 340] is studied in more detail. Let $(\alpha)$ be a partition of $n$, let $q$ be an integer less than $n$, and let $T_{(\alpha)}$ be the diagram of $(\alpha)$. Then associated with $T_{(\alpha)}$ is a skew diagram $T^*_{(\alpha)q}$ which completely describes the $q$-hook structure of $T_{(\alpha)}$; each node of $T^*_{(\alpha)q}$ represents a $q$-hook of $T_{(\alpha)}$ and each $r$-hook of $T^*_{(\alpha)q}$ represents an $rq$-hook of $T_{(\alpha)}$; moreover, to each $q$-chain of $T_{(\alpha)}$ there corresponds a constituent of $T^*_{(\alpha)q}$ and each constituent of $T^*_{(\alpha)q}$ is a right diagram. Let $T_{(\alpha)}$ have the $q$-core $T_{(\alpha_0)}$ consisting of $a$ nodes, where $n=a+bq$, let $\chi_q{}^*$ denote the number of standard skew diagrams belonging to the shape $T^*_{(\alpha)q}$, let $P_b$ be a product of $b$ cycles each of length $q$ on the last $bq$ letters and $V$ any permutation on the first $a$ letters. Then the value of $\chi$ for $VP_b$ is $\pm\chi_q{}^*$ times the value of $\chi_{(\alpha_0)}$ for $V$. This result is of central importance in the proof of Nakayama's conjecture on the two-sided ideals of the modular group algebra of $S_n$ [see the following review]. The present paper concludes with a derivation of some identities connecting the degrees of the irreducible representations of $S_n$. *R. M. Thrall* (Ann Arbor, Mich.).

**Brauer, Richard. On a conjecture by Nakayama.** Trans. Roy. Soc. Canada. Sect. III. (3) **41**, 11–19 (1947).

**Robinson, G. de B. On a conjecture by Nakayama.** Trans. Roy. Soc. Canada. Sect. III. (3) **41**, 20–25 (1947).

Let $G$ be a finite group whose irreducible characters in the complex field are $\zeta_1, \cdots, \zeta_k$. Let $p$ be a prime number and let $\Gamma$ be the group algebra of $G$ over a field of characteristic $p$. A two-sided ideal of $\Gamma$ is called a $p$-block if it has a unit element and if it cannot be written as a direct sum of two-sided ideals. Each character $\zeta_i$ is associated with a unique block to which it is said to belong; one of the important questions in representation theory is the determination of which characters $\zeta_i$ belong to the same block. For $G=S_n$, the symmetric group of degree $n$, each irreducible character is associated with a partition $(\lambda)$ of $n$ and is denoted by $\zeta_{(\lambda)}$. Associated with each partition $(\lambda)$ of $n$ is a partition $(\lambda^*)$ of $n-pb(\lambda)$ called the $p$-core of $(\lambda)$. The concept of $p$-core was introduced by T. Nakayama [Jap. J. Math. **17**, 165–184; 411–423 (1941); these Rev. **3**, 195, 196; **4**, 340] who conjectured that two characters $(\lambda)$, $(\mu)$ belong to the same block if and only if $(\lambda^*)=(\mu^*)$. The authors of the present papers have now proved this conjecture. In Brauer's paper certain properties of the defect group of a block are listed first for an arbitrary group and then in detail for the symmetric group. It is proved that the number of $p$-blocks for $S_n$ is equal to the number of $p$-cores of partitions of $n$. Nakayama's complete conjecture is next shown to be a consequence of three properties of the characters $\zeta_{(\lambda)}$. These three properties are then established in Robinson's paper. The first of these properties is that $\zeta_{(\lambda)}$ vanishes for all permutations having more than $b(\lambda)$ $p$-cycles, and the second is that there exist permutations with exactly $b(\lambda)$ $p$-cycles for which $\zeta_{(\lambda)}$ does not vanish. These first two properties follow fairly easily from Nakayama's results, although Robinson's proof is not along those lines. The third property is a congruence satisfied by the characters $\chi_{(\lambda)}$ for all permutations having $b(\lambda)$ $p$-cycles and no other cycles of length divisible by $p$. Its proof goes much deeper into the theory of characters of the symmetric group and represents a culmination point for Robinson's series of papers, of which the third is reviewed above.

*R. M. Thrall* (Ann Arbor, Mich.).

**Hayashida, Tsuyoshi. Arc-wise connected subgroup of a vector group.** Kōdai Math. Sem. Rep., no. 1, 16, 19 (1949).

**Homma, Tatsuo, and Minagawa, Takizo. Vector-group in real Euclidean space.** Kōdai Math. Sem. Rep., no. 1, 19–20 (1949).

Let $R_n$ denote the real $n$-dimensional vector group. These two notes prove that any arcwise connected subgroup of $R_n$ is a real vector subgroup. The authors state that this result has also been proved by Iwamura and Kuranishi.

*D. Montgomery* (Princeton, N. J.).

**Gleason, Andrew M. On the structure of locally compact groups.** Proc. Nat. Acad. Sci. U. S. A. **35**, 384–386 (1949).

Voici les principaux résultats annoncés. Théorème 1: soit $G$ un groupe topologique, et supposons que $G$ possède un sous-groupe invariant fermé $N$ tel que $N$ et $G/N$ soient des groupes de Lie; $G$ est alors lui-même un groupe de Lie. Théorème 6: tout groupe localement compact possède une suite de composition finie telle que tous les groupes quotients soient abéliens, ou compacts, ou totalement discontinus, ou simples. Théorème 7: tout groupe localement compact connexe contient un sous-groupe invariant résoluble maximal; le groupe quotient correspondant est semi-simple, c'est-à-dire ne contient aucun sous-groupe résoluble invariant autre que $\{e\}$. Théorème 8: tout groupe localement compact connexe semi-simple est un produit direct de

groupes indécomposables; tous les facteurs sauf un nombre fini sont des groupes de Lie compacts simples; les autres facteurs sont métrisables. Il est évidemment à peine nécessaire de préciser que ces résultats constituent le premier pas véritablement sérieux vers une solution du cinquième problème de Hilbert (dont l'étude, en vertu des résultats que l'on connait déja, se trouve ramenée à celle des groupes simples). *R. Godement* (Nancy).

**Iwasawa, Kenkichi. On some types of topological groups.** Ann. of Math. (2) **50**, 507–558 (1949).

§ 1. Soient $K$ un groupe compact, $A(K)$ son groupe d'automorphismes, $I(K)$ le groupe des automorphismes intérieurs de $K$; si l'on munit ces groupes de la topologie usuelle, $A(K)/I(K)$ est un groupe totalement discontinu (la démonstration consiste, en passant par l'intermédiaire des représentations irréductibles de $K$, à se ramener au cas des groupes de Lie: les résultats connus sur la structure de ceux-ci montrent alors que $A(K)/I(K)$ est discret). Supposons maintenant que $K$ soit un sous-groupe invariant compact d'un groupe connexe $G$; alors les automorphismes intérieurs de $G$, considérés sur $K$, sont dans $I(K)$; conséquences: (a) on a $G=HK$ où $H$ est le groupe des $s \in G$ qui permutent aux $x \in K$; (b) tout sous-groupe de $K$, invariant dans $K$, l'est aussi dans $G$; (c) si $K$ est abélien, il appartient au centre de $G$.

§ 2. Soient $G$ un groupe topologique, et $D_1(G)$ le sous-groupe fermé engendré par les commutateurs des éléments de $G$; en itérant $D_1$, on fabrique $D_2(G), \cdots$; on pose $D_\alpha(G)=\bigcap_\alpha D_\alpha(G)$, ce qui permet de continuer la construction transfiniment. On obtient ainsi la série des sous-groupes commutateurs de $G$; cette série contient un élément minimal: si celui-ci est $\{e\}$, on dit que $G$ est résoluble. En s'appuyant sur le § 1, on montre facilement que la série en question est finie si $G$ est localement compact connexe; on a même $D_1(G)=D_2(G)$ si $G$ est compact et connexe.

§ 3. (a) Soit $G$ un groupe de Lie connexe arbitraire; alors les sous-groupes compacts maximaux de $G$ sont deux à deux conjugués; si de plus $K$ est l'un d'entre eux, il existe $r$ sous-groupes à un paramètre $H_1, \cdots, H_r$ de $G$ tels que tout $s \in G$ admette une décomposition (unique et continue) $s=h_1 \cdots h_r k$ ($h_i \in H_i$; $k \in K$). Ce résultat se trouve déjà dans A. Malcev [Rec. Math. [Mat. Sbornik] N.S. **16(58)**, 163–190 (1945); ces Rev. **7**, 115]. Pour le démontrer, l'auteur établit plusieurs lemmes; citons celui-ci: soit $G$ un groupe localement compact possédant un sous-groupe invariant $N$, isomorphe à $R^n$, tel que $G/N$ soit compact; alors il existe un sous-groupe compact $K$, déterminé à un automorphisme intérieur près de $G$, tel que $G=KN$, $K \cap N=\{e\}$ (la démonstration de ce lemme repose sur un emploi original de la mesure de Haar); naturellement, la théorie des groupes semi-simples joue un grand rôle dans la démonstration du théorème en question. (b) Si un groupe localement compact $G$ possède un sous-groupe invariant $N$ tel que $N$ et $G/N$ soient des groupes de Lie, $G$ est lui-même un groupe de Lie. La démonstration consiste à se ramener au cas où $N=R^n$, $G/N$ étant connexe; on montre alors l'existence dans $G$ d'une "section modulo $N$" qui est homéomorphe à $G/N$, soit $f(s)$ ($s \in G/N$); on a alors des relations $f(s)f(t)=u(s,t)f(st)$, où $u(s,t) \in N$; en "régularisant" $u$ au moyen de fonctions différentiables sur $G/N$, on peut s'arranger pour que $u(s,t)$ soit continûment différentiable, d'où résulte facilement la propriété annoncée.

§ 4. Un groupe localement compact $G$ est un $(L)$-groupe s'il contient une famille $N_\alpha$ de sous-groupes invariants tels que les $G/N_\alpha$ soient des groupes de Lie, l'intersection des $N_\alpha$ étant réduite à $\{e\}$. Un $(L)$-groupe connexe contient exactement un sous-groupe invariant compact maximal $K$, et $G/K$ est un groupe de Lie. Soit $N$ un sous-groupe invariant d'un groupe localement compact connexe $G$, tel que $N$ et $G/N$ soient des $(L)$-groupes; alors $G$ est un $(L)$-groupe. Tout groupe localement compact connexe et résoluble [cf. § 2] est un $(L)$-groupe. Soit $G$ un $(L)$-groupe connexe; pour tout voisinage $U$ de l'unité dans $G$, il existe un sous-groupe invariant compact $K \subset U$, et un groupe de Lie local $L \subset G$, tels que $G$ soit localement isomorphe à $L \times K$ (ceci implique la solution du cinquième problème de Hilbert pour les $(L)$-groupes). Enfin, le théorème (a) du § 3 est valable pour les $(L)$-groupes connexes.

§ 5. Ce § contient des résultats sur les groupes localement compacts. Pour un tel groupe $G$, il existe un sous-groupe invariant, compact et connexe, qui contient tous les sous-groupes invariants, compacts et connexes de $G$. Si $G$ est connexe, on a une propriété analogue en remplaçant "compact connexe" par "résoluble," ou par "résoluble connexe"; dans ce dernier cas, le sous-groupe maximal obtenu est appelé le radical de $G$, soit $N_0$. On dit que $G$ est un $(C)$-groupe connexe si $G$ est connexe et si $G/N_0$ est compact; on dit que $G$ est un $(C)$-groupe s'il possède un système "complet" de sous-groupes invariants $N_\alpha$ tels que la composante connexe de l'unité dans $G/N_\alpha$ soit un $(C)$-groupe connexe de Lie: cette définition est en accord avec la précédente dans le cas où $G$ est connexe. L'auteur énonce pour les $(C)$-groupes des résultats analogues à ceux du § 4. Dans tout groupe localement compact connexe $G$, il existe un sous-groupe invariant maximal unique du type $(L)$-groupe; de même, parmi les sous-groupes invariants $H$ tels que $G/H$ soit un $(L)$-groupe, il en existe un qui est plus petit que tous les autres.

L'auteur termine en énonçant quelques conjectures (par exemple: tout groupe localement compact connexe est un $(L)$-groupe), et annonce leur équivalence avec celle-ci: un groupe simple connexe et localement compact est un groupe de Lie. Une hypothèse identique est faite par A. M. Gleason [voir l'analyse ci-dessus] qui semble du reste avoir retrouvé des résultats analogues à ceux du présent travail.

*R. Godement* (Nancy).

**Vilenkin, N. Ya. Fibered Abelian topological groups and their character theory.** Mat. Sbornik N.S. **24(66)**, 189–226 (1949). (Russian)

In earlier papers [Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 311–340 (1946); Mat. Sbornik N.S. **22(64)**, 135–177 (1948); these Rev. **8**, 312; **9**, 497] the author has introduced and studied the notions of coseparability and weak separability for topological Abelian groups. He now defines a topological Abelian group to be fibered if $K(G)$, the component of the identity, is locally compact, satisfies the second axiom of countability, and is such that $G/K(G)$ is locally weakly separable. Here a group is locally weakly separable if it admits a weakly separable open subgroup with a countable quotient.

The author's discussion of fibered groups falls naturally into four parts. Sections 1 and 2 deal with various elementary properties of fibered groups and the related "quasi discrete" and "quasi compact" groups. Typical theorems: Every closed subgroup of a fibered group is a fibered group. Every fibered group is the direct sum of a vector subgroup and a fibered group whose component of the identity is compact. A continuous homomorphism of a fibered group

on a fibered group is open. Section 3 deals with an extension of the Pontrjagin duality theory to fibered groups. The group of characters of a fibered group is a fibered group and the group of characters of the group of characters is isomorphic to the original group in a natural way. In section 4 the author considers pairs $G$, $T$ consisting of a fibered group $G$ which is the union of its compact subgroups and a discrete torsion-free group $T$. Conditions are obtained under which every fibered extension of $G$ by $T$ is the direct sum of $G$ and $T$. This problem for discrete groups has been considered by Baer [Ann. of Math. (2) **37**, 766–781 (1936)]. The dual problem in which $G$ is compact, connected and separable and $T$ is zero dimensional is also discussed. In the final section universal groups of various kinds are considered.

*G. W. Mackey* (Nancy).

**Vilenkin, N. Ya. On the theory of lacunary orthogonal systems.** Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 245–252 (1949). (Russian)

Let $X$ be a countable Abelian torsion group and let $G$ be its compact character group. Then the members of $X$ constitute a complete orthonormal set of functions on $G$. The "lacunary" systems considered by the author are subsets $\psi$ of $X$ constructed in the following manner. Choose an ascending sequence $X_0 \subset X_1 \subset X_2 \subset \cdots$ of subgroups of $X$ such that $X_0$ is the identity, $X_{n+1}/X_n$ is cyclic of prime order and $X = \bigcup_{n=1} X_n$. Choose a single element $\psi_n$ from each $X_n - X_{n-1}$. Let $\psi = \{\psi_0, \psi_1, \psi_2, \cdots\}$. The bulk of the present paper is devoted to a proof of the following generalization of a known theorem for the Rademacher functions. If the series $\sum_{n=0}^{\infty} c_n$ converges then $\sum_{n=0}^{\infty} c_n \psi_n$ belongs to $L^q$ for all $q$. When $X$ is the direct product of countably many replicas of the group of order two then almost all of $G$ may be mapped onto almost all of the unit interval in such a manner that the members of $X$ become the Walsh functions and the members of $\psi$ the Rademacher functions. A number of other theorems about the Rademacher functions are similarly generalized to theorems about the systems $\psi$. Detailed proofs are for the most part omitted.

*G. W. Mackey* (Nancy).

**Leray, Jean. Espace où opère un groupe de Lie compact et connexe.** C. R. Acad. Sci. Paris **228**, 1545–1547 (1949).

Let $G$ be a compact connected Lie group, $X$ a locally compact space in which $G$ operates by transformations $x \to \psi(g, x)$. Let $H_X$ be the cohomology ring of $X$ over a commutative field $C$ of characteristic zero. The author sketches a construction by which there can be associated with $G$ a vector space $\Delta$ (over $C$) of dimension equal to rank $G$. The elements of $\Delta$ are certain differentials $\delta$. A differential $\delta$ here means any system of linear endomorphisms of $H_X$, one for each $(X, \psi)$ having certain formal properties. The exterior algebra $\Lambda\Delta$ of $\Delta$, generated by compositions of elements of $\Delta$, is isomorphic to the homology ring of $G$. A degree $-q>0$ is attached to certain elements of $\Delta$; these elements lower the degrees of elements of $H_X$ by $q$ units. If $G$ operates in $X$ and $Y$ and if $\xi$ is a mapping $X \to Y$ which commutes with each $g \in G$, the homomorphism $\xi^{-1}: H_Y \to H_X$ commutes with each $\delta \in \Delta$. A special case is $X = G$, $\xi$ given by $g \to \psi(g, y)$; the mapping $\xi^{-1}: H_Y \to H_G$ is independent of $y$. It is shown that $H_Y$ can be expressed as a tensor product $H_Y'' \otimes \Lambda H_Y'$ such that $\xi^{-1}$ maps $\Lambda H_Y'$ isomorphically into $H_G$ and annuls all elements of $H_Y''$ of positive degree.

*P. A. Smith* (New York, N. Y.).

**Borel, Armand. Some remarks about Lie groups transitive on spheres and tori.** Bull. Amer. Math. Soc. **55**, 580–587 (1949).

This paper generalizes and sharpens earlier results by Montgomery and Samelson concerning compact Lie groups transitive on a sphere or torus [Bull. Amer. Math. Soc. **49**, 455–456 (1943); Ann. of Math. (2) **44**, 454–470 (1943); these Rev. **4**, 250; **5**, 60]. Some of the results are stated below where $G$ is always a compact connected Lie group. If $G$ acts transitively on an $n$-dimensional homogeneous space $W$ and if for some $j$ $(1 \leqq j \leqq n-1)$ the $j$th Betti number equals the binomial coefficient $C_{nj}$ then $W$ is homeomorphic to the $n$-dimensional torus. If $G$ acts transitively and effectively on a simply connected space $W$ which has an Euler characteristic equal to a prime number then $G$ is simple. The only $G$ acting transitively on $S^{2r}$ is locally isomorphic to $B_r$ and also, when $r=3$, to $G_2$. The even-dimensional spheres are the only simply connected spaces of characteristic two on which $G$ can act transitively. For each prime $p>2$ there are only a finite number of simply connected spaces of characteristic $p$ on which $G$ can operate and these are classified.

*D. Montgomery* (Princeton, N. J.).

**Poliščuk, E. M. On the exponential representation of the elements of a semisimple complex Lie group.** Mat. Sbornik N.S. **24(66)**, 237–248 (1949). (Russian)

Let $G$ be a complex semisimple Lie group, $A$ its adjoint group, and $L(A)$ the latter's Lie algebra. The author's main theorem asserts that any $a \in A$ can be written in the form $a = be^u$, where $b \in A$, $u \in L(A)$, $b$ and $e^u$ commute, and $b^m = 1$ with $m$ independent of $a$. The proof makes heavy use of the results and methods of Gantmacher [Rec. Math. [Mat. Sbornik] N.S. **5(47)**, 101–146 (1939); these Rev. **1**, 163]. From this theorem it follows that, for a suitable integer $n$, all $n$th powers of elements of $G$ are representable as exponentials. The smallest value for this integer $n$ is examined for each of the simple groups (including the exceptional ones).

*I. Kaplansky* (Chicago, Ill.).

**Toyama, Hiraku, and Kuranishi, Masatake. A note on generators of compact Lie groups.** Kōdai Math. Sem. Rep., no. **1**, 17–18 (1949).

The following theorem was proved by Auerbach [Studia Math. **5**, 43–49 (1935)]. Let $G$ be a connected compact Lie group. For two elements $x$ and $y$ of $G$ and an integer $k$ let $M(x, y, k)$ be the set of elements $p = \prod_{i=1}^{k} v_i$, where $v_i = x^{n_i}$ when $i$ is odd and $v_i = y^{n_i}$ when $i$ is even. Let $M(x, y)$ be the union for $k = 1, 2, \cdots$ of $M(x, y, k)$. Then there exist $x, y$ so that $M(x, y)$ is dense in $G$. It is shown here that for each such $G$ there is a $k$ such that $M(x, y, k)$ is dense in $G$. If $f(G)$ is the minimum of such $k$'s, it is proved that $f(G) \geqq \dim G / \text{rank } G$.

*D. Montgomery.*

**Toyama, Hiraku. On discrete subgroups of a Lie group.** Kōdai Math. Sem. Rep., no. **2**, 36–37 (1949).

Let $G$ be a topological group and $g_1, g_2, \cdots$ a sequence of discrete subgroups. The group $G$ is said to be approximated by this sequence of discrete subgroups provided that for any open subset $U$ of $G$ all $g_i$, except a finite number, have a nonempty intersection with $U$. The main result is the following theorem. If $G$ is a connected Lie group with discrete center, then $G$ cannot be approximated by a sequence of discrete subgroups. Two examples are given of noncommutative Lie groups with nondiscrete center, one of which can be approximated and the other cannot. It is remarked

that a noncommutative compact Lie group cannot be approximated by finite subgroups. *D. Montgomery.*

**Gotô, Morikuni. Faithful representations of Lie groups. I.** Math. Japonicae 1, 107–119 (1948).

The author considers the problem of determining those Lie groups which are isomorphic (in the large) to linear Lie groups (such isomorphisms are known always to exist locally). The main results are to appear in part II. The present paper prepares the ground by treating the question of recognizing those Lie subgroups of a given Lie group which are closed. The author states that a number of his results are closely related to the work of others, particularly A. Malcev. By the radical of a Lie group $G$ is meant the Lie subgroup generated by the maximal solvable ideal of the Lie algebra of $G$. Theorems: (1) The closure $\bar{H}$ of a Lie subgroup of a Lie group $G$ equals $AH$, where $A$ is the radical of any maximal Lie subgroup $K$ of $\bar{H}$; the closures of $K \bigcap H$ and $A \bigcap H$ are $K$ and $A$, respectively; $H$ is closed if and only if it contains the closures of its 1-parameter subgroups. (2) Let $H$ be a Lie subgroup of a Lie group $G$ and suppose that $H$ contains a maximal semisimple Lie subgroup with finite center. Then $H$ is closed if and only if its radical is closed in $G$. *P. A. Smith* (New York, N. Y.).

## NUMBER THEORY

**Gloden, A. Binomial factorisations.** Scripta Math. 14, 283 (1948).

A résumé of the author's results on the factors of $x^4+1$. These results give 88 new factorisations of numbers of this form which are less than $10^{12}$. *D. H. Lehmer.*

**Katz, Alexander. Some more new factors of Fibonacci-numbers.** Riveon Lematematika 3, 14 (1949). (Hebrew)

The prime factors 29717, 151549, 6263, 128621, and 119809 are announced for $u_{117}$, $v_{73}$, $v_{109}$, $v_{119}$, and $v_{128}$, respectively. No other new factors of Fibonacci's $u_n$ and $v_n = u_{2n}/u_n$ exist below 200000. [See Jarden, same Riveon 1, 35–37 (1946); 2, 22 (1947); Jarden and the author, ibid., 35 (1948); these Rev. 8, 135; 9, 134, 411]. *D. H. Lehmer.*

**Jarden, Dov. Disjunctive sequences.** Riveon Lematematika 3, 15–18 (1949). (Hebrew)

A disjunctive sequence is a sequence of integers every two of which are relatively prime. Two methods for the construction of such sequences are given and compared. *A. Dvoretzky* (Princeton, N. J.).

**Jarden, Dov. Table of the distribution of zeros in the period mod $p$ of a recurring sequence of order 3.** Riveon Lematematika 2, 65–66 (1948). (Hebrew)

This note gives numerical data on the residues mod $p$ of the terms of the following recurring series:

$$\begin{array}{llll} U_n = U_{n-2} + U_{n-3}; & U_0 = 0, & U_1 = 0, & U_2 = 1; \\ V_n = V_{n-2} + V_{n-3}; & V_0 = 3, & V_1 = 0, & V_2 = 2; \\ \bar{U}_n = -\bar{U}_{n-2} + \bar{U}_{n-3}; & \bar{U}_0 = 0, & \bar{U}_1 = 0, & \bar{U}_2 = 1; \\ \bar{V}_n = -\bar{V}_{n-2} + \bar{V}_{n-3}; & \bar{V}_0 = 3, & \bar{V}_1 = 0, & \bar{V}_2 = -2; \end{array}$$

the values of $p$ considered are the primes less than 37. Besides the periods the author tabulates the ranks of those terms which are divisible by $p$. It is conjectured (correctly) that, except for $p = 23$ and 31, the period of $U_n$ or of $\bar{U}_n$ mod $p$ is some divisor of $p^2-1$ or $p^2+p+1$. The author fails to notice that these exceptional primes are the discriminants of scales of relation of the above recurring series. *D. H. Lehmer* (Berkeley, Calif.).

**Neville, E. H. The structure of Farey series.** Proc. London Math. Soc. (2) 51, 132–144 (1949).

The Farey series $\mathfrak{F}_n$ of order $n$ is the set of fractions $h/k$, with $0 \leqq h \leqq k \leqq n$, $(h, k) = 1$. A detailed study of the structure of such series is given. It is shown how a particular series can be used to best advantage in solving linear Diophantine equations. Also, if $bc - ad = 1$ and $n > \max(b, d)$, then the terms of $\mathfrak{F}_n$ between $a/b$ and $c/d$ are the fractions of the form $(\lambda a + \mu c)/(\lambda b + \mu d)$ for which $\lambda$, $\mu$ are positive relatively prime integers, and $\lambda b + \mu d \leqq n$. The number of such terms is the number of coprime pairs $\lambda$, $\mu$ satisfying the condition $\lambda/(n/b) + \mu/(n/d) \leqq 1$, and is equal to $(3/\pi^2)n^2/bd + O(n \log n)$. Thus for a given $n$ the number of terms in any interval of $\mathfrak{F}_n$ is approximately proportional to the length of the interval. This fact is used in discussing the location of a given fraction either as a term of the series or for telling the terms of the series between which it lies. Finally, problems in the actual compilation of a Farey series are discussed. [Correction. In the example on page 135 the [ ] should be replaced by { }, where $\{x\}$ denotes the integer nearest to $x$.] *W. H. Simons* (Vancouver, B. C.).

**Lehmer, D. H. On the converse of Fermat's theorem. II.** Amer. Math. Monthly 56, 300–309 (1949).

To have a rapid and positive method for identifying a number greater than $10^7$ as a prime, the author published [Amer. Math. Monthly 43, 347–354 (1936)] a list of all composite $n$, $10^7 < n < 10^8$, having no factor greater than 313 and for which $n \mid 2^n - 2$, every $n$ with its smallest factor. It contains 526 entries and was based on the extant table of least exponents $e$ of $p < 300\,000$. This table was inadequate both in extent and in accuracy [see the comments by the author [Math. Tables and Other Aids to Computation 2, 313 (1947)] on Kraïtchik, Recherches sur la Théorie des Nombres, v. 1, Paris, 1924] for an extension of the list of $n$. The author was connected with the early operations of the ENIAC and could use this electronic computor for the calculation of the requisite numbers $e$. He gives a description of the "setup" of the ENIAC, showing how this modern computing device using a method differing greatly from the one used by human computers solves such questions with incredible speed. The table of exponents $e$ furnished by the ENIAC suffices to extend the list of $n$ up to $10^9$. The article gives it up to $2 \cdot 10^8$, which, given the table of $e$, still required a considerable amount of work. The practical method of performing the operations to be done to identify a number (between $10^7$ and $2 \cdot 10^8$) as a prime is explained in the article of 1936. To the present article, the author has added a proof of the existence of an infinity of numbers $n$, which are products of three factors and satisfy $n \mid 2^n - 2$. *N. G. W. H. Beeger* (Amsterdam).

**Palamà, Giuseppe. Saggio di una nuova trattazione delle multigrade.** Boll. Un. Mat. Ital. (3) 3, 263–278 (1948).

An expository article giving known results and numerical examples in the problem, known as the Tarry-Escott problem, of finding two sets of integers $a_1, \cdots, a_p$ and $b_1, \cdots, b_q$

such that

$$(1) \qquad \sum_{i=1}^{p} a_i^k = \sum_{i=1}^{q} b_i^k, \qquad k=(m_1, \cdots, m_r),$$

where the $m_i$ are positive integers. Mention is made of the application of multigrade equations to (a) Waring's problem, (b) the calculation of logarithms, (c) equations having unity as a multiple root and (d) the theory of irreducible polynomials. The paper contains copious references.

*W. H. Simons* (Vancouver, B. C.).

**Palamà, Giuseppe. Un teorema analogo a quello di Tarry. Osservazioni su altri noti. Applicazioni.** Atti Sem. Mat. Fis. Univ. Modena **2**, 116–142 (1948).

This paper is concerned with the so-called Tarry-Escott problem of equal sums of like powers. The theorem mentioned in the title is the following. If $\sum_{i=1}^{p} a_i^k = \sum_{i=1}^{p} b_i^k$ $(k=0, 1, \cdots, n)$, then

$$\sum_{i=1}^{p} a_i^k + (s-c_i)^k = \sum_{i=1}^{p} b_i + (s-d_i)^k, \qquad k=0, 1, \cdots, n+1,$$

where $s$ is an arbitrary parameter and $(c_i, d_i)=(a_i, b_i)$ or $(b_i, a_i)$ according as $n$ is even or odd. The theorem is illustrated by numerous numerical examples.

*D. H. Lehmer* (Berkeley, Calif.).

**Postnikov, A. G. On the differential independence of Dirichlet series.** Doklady Akad. Nauk SSSR (N.S.) **66**, 561–564 (1949). (Russian)

Let $m$ be a positive integer and $\chi$ any character modulo $m$. It is shown that the $\varphi(m)$ Dirichlet series $L(s, \chi)$ are differentially independent. By this is meant that no nonzero polynomial in $s$, the series and their derivatives of arbitrary orders, with constant coefficients, vanishes for every $s$. The author proves first that if several Dirichlet series of the type $\sum a_n n^{-s}$ are differentially dependent, then the series obtained from them by suppressing all terms except those in which $n$ is prime, and sufficiently large, satisfy together a linear differential equation.

*J. F. Ritt.*

**Popken, J. Remark on my paper "On Lambert's proof for the irrationality of $\pi$."** Nederl. Akad. Wetensch., Proc. **52**, 504 = Indagationes Math. **11**, 164 (1949).

The author acknowledges that the method of his paper [same Proc. **43**, 712–714 (1940); these Rev. **2**, 149] is nearly the same as that of Hermite [Cours de la faculté des sciences . . . , 4th ed., Hermann, Paris, 1891, pp. 74–75].

**Gel'fond, A. O. On the algebraic independence of algebraic powers of algebraic numbers.** Doklady Akad. Nauk SSSR (N.S.) **64**, 277–280 (1949). (Russian)

Let $\alpha$ be a root of an irreducible algebraic equation of degree 3 and $a \neq 0, 1$ any algebraic number. The author shows that, for any fixed interpretation of $\log a$, the numbers $a^{\alpha} = e^{\alpha \log a}$ and $a^{\alpha^2} = e^{\alpha^2 \log a}$ are algebraically independent in the field of the rational numbers. The proof, which is not given in full detail, makes use of results said to have been proved by the author in another paper inaccessible to the reviewer. It is stated that the method can be applied without appreciable change to prove that if $\omega_1, \omega_2, \cdots, \omega_r$ ($\omega_1$ rational) form a basis for the ring of algebraic integers of an algebraic field $K$, then no relation $P(a^{\omega_i}, a^{\omega_j})=0$ $(i \neq j \neq 1)$ is possible where $P(x, y)$ is a polynomial in $x, y$ with rational coefficients; an improvement of this result, which is to appear elsewhere, is announced. [Note. The reviewer suspects that in lemma 3 the case $k_1=k_2=k_3=0$ is meant to be excluded. Otherwise the lemma becomes trivial since $B_r$ is then a multiple of $f^{(s)}(0)$ which is zero by hypothesis.]

*R. A. Rankin* (Cambridge, England).

**Gel'fond, A. O. On some general cases of the distribution of the fractional parts of functions.** Doklady Akad. Nauk SSSR (N.S.) **64**, 437–440 (1949). (Russian)

Let $L$ be an increasing (in the wide sense) sequence of real numbers $y_1, y_2, \cdots$ and $M$ a sequence of points $\tau_k=(\beta_{1,k}, \beta_{2,k}, \cdots, \beta_{r,k})$ in $r$-dimensional space everywhere dense in the unit cube $0 \leqq \beta_{i,k} \leqq 1$ $(1 \leqq i \leqq r,\ k=1, 2, \cdots)$. The fractional parts $\{f_i(y)\}$ of the $r$ functions $f_i(y)$ are said to be $(\varphi, M)$ everywhere densely distributed if the inequalities

$$0 \leqq \{f_i(y)\} - \beta_{i,k} \leqq \varphi(y) \qquad i=1, 2, \cdots, r,$$

where $\varphi(y)$ is any given monotonically decreasing function satisfying $0 \leqq \varphi(y) \leqq 1$ and $\lim_{y \to \infty} \varphi(y)=0$, possess for every point $\tau_k$ of $M$ an infinity of solutions in values of $y$ belonging to $L$. The author proves the following two theorems. (I) Let $f_1(t_1, y), f_2(t_2, y), \cdots, f_r(t_r, y)$ satisfy the following three conditions. (1) The functions $f_i(t_i, y)$ are defined for all $y \in L$ in the segment $a_i \leqq t_i \leqq b_i$ $(1 \leqq i \leqq r)$. (2) For all $i, k$, $\partial f_i / \partial t_j = f_i'(t_i, y_k)$ is continuous and increasing in the segment $[a_i, b_i]$ and $f_i'(a_i, y_k) \geqq 1$. (3) For fixed $t_i$ in the segment $[a_i, b_i]$, $f_i'(a_i, y_k)$ tends monotonically to infinity with $k$ so that $f_i(t_i, y_k) \to \infty$ as $k \to \infty$ $(1 \leqq i \leqq r)$. Then, for any fixed sequence $M$, arbitrary $\epsilon > 0$ and $c_i, d_i$ such that $a_i \leqq c_i < c_i + \epsilon = d_i \leqq b_i$, and for any given function $\varphi(y)$ of the type defined above, there exists at least one point $(\alpha_1, \alpha_2, \cdots, \alpha_r)$ with $c_i \leqq \alpha_i \leqq d_i$ $(1 \leqq i \leqq r)$ such that the $r$ functions $f_i(\alpha_i, y)$ are $(\varphi, M)$ everywhere densely distributed.

(II) Suppose that, in addition to (1), (2) and (3) of the first theorem,

$$\lim_{k \to \infty} \frac{f_i'(a_i, y_{k+1})}{f_i'(b_i, y_k)} \varphi(y_k) = \infty, \qquad 1 \leqq i \leqq r.$$

Also let $M$ be an arbitrary infinite sequence of points $\tau_k$ in the $r$-dimensional unit cube consisting, possibly, of only a finite number of distinct points. Also let $\varphi(y)$ be any given function such that $0 \leqq \varphi(y) \leqq 1$ ($\varphi(y)=1$ is allowed). Then, in any segment $[c_i, d_i]$ inside $[a_i, b_i]$ there is at least one point such that

$$\lim_{k \to \infty} [\{f_i(\alpha_i, y_k)\} - \beta_{i,k}] / \varphi(y_k) = 0.$$

Examples, for $r=1$, of functions $f(t, y)$ satisfying the conditions of the theorems are given and it is shown that (II) can be used to prove Fabry's theorem on gap series.

*R. A. Rankin* (Cambridge, England).

**Revuz, André. Sur la répartition des points $e^{n\omega}$.** C. R. Acad. Sci. Paris **228**, 1466–1467 (1949).

The note contains results concerning the distribution of the numbers $\omega, 2\omega, 3\omega, \cdots \pmod 1$, expressed in terms of the continued fraction for the irrational number $\omega$. They are, I believe, all known in substance, though perhaps not in the form given here.

*H. Davenport* (London).

**Varnavides, P. On the quadratic form $x^2-7y^2$.** Proc. Roy. Soc. London. Ser. A. **197**, 256–268 (1949).

Es wird die Frage nach der Approximation zweier inhomogener Linearformen der Form $x+y\sqrt{7}-a$ und $x-y\sqrt{7}-b$ durch ganze rationale Zahlen $x, y$, wobei $a, b$ beliebige reelle Grössen sind, durch folgenden Satz beantwortet. Für beliebige reelle $a, b$ gibt es ganze Zahlen $\xi$ aus $K(\sqrt{7})$,

die die Ungleichung $|(\xi-a)(\xi'-b)| \leqq \frac{9}{7}\varepsilon$ befriedigen, wenn $\xi'$ die Konjugierte von $\xi$ ist. Sind $a$, $b$ speziell von der Form $a=\omega/\alpha_0+\xi_0$, $b=\omega'/\alpha_0'+\xi_0'$, wo $\alpha_0=\frac{1}{2}(7+5\sqrt{7})$, $\alpha_0'=\frac{1}{2}(7-5\sqrt{7})$, $\omega$ irgendeine Einheit von $K(\sqrt{7})$ und $\xi_0$ eine ganze Zahl aus $K(\sqrt{7})$ bedeuten, so gibt es unendlich viele Lösungen obiger Ungleichung, für die sogar das Gleichheitszeichen eintritt, aber keine Lösungen mit dem Kleinerzeichen. Wenn aber $a$ und $b$ nicht diese spezielle Gestalt haben, gibt es eine ganze Zahl $\xi$ von $K(\sqrt{7})$, die die schärfere Ungleichung

$$|(\xi-a)(\xi'-b)| \leqq \frac{1}{1.56} < \frac{9}{14}$$

erfüllt. Für den Körper $K(\sqrt{11})$ hat der Verf. in seiner Dissertation [London, 1948] bereits ähnliche Resultate gezeigt. Der Beweis gründet sich auf eine Davenportsche Methode [Nederl. Akad. Wetensch., Proc. **50**, 378–389 = Indagationes Math. **9**, 236–247 (1947); diese Rev. **8**, 565].

*T. Schneider* (Göttingen).

**Hlawka, Edmund. Über eine Verallgemeinerung des Satzes von Mordell.** Ann. of Math. (2) **50**, 314–317 (1949).

The following result was proved by a method of Siegel [Davenport, Acta Arith. **2**, 262–265 (1937)]. For any lattice in $n$-dimensional space, of determinant 1, there exist positive numbers $\lambda_1, \cdots, \lambda_n$, with

$$\lambda_1 \cdots \lambda_n \geqq (1 \cdot 2^2 \cdot 3^3 \cdot \cdots \cdot (n-1)^{n-1} n!)^{-1}$$

such that the "box" defined by $|x_1| < \lambda_1, \cdots, |x_n| < \lambda_n$ contains no lattice point except the origin $O$. The author gives another result of a similar kind, which he enunciates as follows. Let $f(x)$ be the distance-function of a convex body, and $A$ a matrix of determinant 1. Then there exists an integral unimodular matrix $U$, independent of $A$, and a diagonal matrix $D$, such that the body $f(UDU^{-1}Ax) \leqq t$ contains no point with integral coordinates [except $O$] if its volume is less than $2^n(n!)^{-4n^2-n-1}$. The proof is complicated, and not easy to follow.

*H. Davenport* (London).

**Rogers, C. A. The successive minima of measurable sets.** Proc. London Math. Soc. (2) **51**, 440–449 (1949).

Let $S$ be any set of points in $n$-dimensional space, of positive Lebesgue measure $V(S)$. Let $\mathbf{D}S$ denote the difference set, consisting of all points $(x_1-y_1, \cdots, x_n-y_n)$, where $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ are points of $S$. With $\mathbf{D}S$, the author associates successive minima $\nu_1, \cdots, \nu_n$, in the sense of Minkowski, defining $\nu_k$ to be the lower bound of all positive numbers $\nu$ for which the set $\mathbf{D}\nu S$ contains $k$ linearly independent points with integral coordinates. One result which he proves is that $\nu_1 \cdots \nu_n V(S) \leqq 2^{\frac{1}{2}(n-1)}$. Some of the results relate to alternative definitions of the successive minima. The paper [submitted in 1946] has now been largely superseded by later work of Rogers [Nederl. Akad. Wetensch., Proc. **52**, 256–263 = Indagationes Math. **11**, 71–78 (1949); these Rev. **10**, 511] and of Chabauty [C. R. Acad. Sci. Paris **227**, 747–749 (1948); **228**, 796–797 (1949); these Rev. **10**, 511]. See also V. Jarník [Časopis Pěst. Mat. Fys. **73**, 9–15 (1948); these Rev. **10**, 102].

*H. Davenport* (London).

**Rédei, L. Vereinfachter Beweis des Satzes von Minkowski-Hajós.** Acta Univ. Szeged. Sect. Sci. Math. **13**, 21–35 (1949).

The author considers the following four theorems. (I) Let $R_n$ be the module of all real vectors $(x)=(x_1, \cdots, x_n)$ and suppose that there is a submodule $U$ of $R_n$ such that there is exactly one element $(x)$ with $0 \leqq x_i < 1$ $(i=1, \cdots, n)$ in every residue class of $R_n/U$. Then $U$ contains at least one element $(0, \cdots, 0, 1, 0, \cdots, 0)$. (Ia) Let $L_i(x)$ $(i=1, \cdots, n)$ be $n$ real linear forms in $x_1, \cdots, x_n$ with determinant $\pm 1$. If the system of inequalities $|L_i(x)| < 1$ $(i=1, \cdots, n)$ has no integral solution other than $x_1 = \cdots = x_n = 0$, then at least one of the forms $L_i(x)$ has integral coefficients. (Ib) Let $\mathfrak{P}$ be a lattice and suppose that, for every given point $(x_1, \cdots, x_n)$, there is one and only one point $(\xi_1, \cdots, \xi_n)$ of $\mathfrak{P}$ such that $0 \leqq x_i - \xi_i < 1$ $(i=1, \cdots, n)$. Then $\mathfrak{P}$ contains a unit vector (i.e., two points whose distance apart is unity). (Ic) If the finite Abelian group $G$ can be represented in the form $G=[\alpha_1] \cdots [\alpha_n]$, where the factors $[\alpha_i]$ are simplexes, then at least one factor is a group. By a simplex $[\alpha]$ is meant a complex of elements $1, \alpha, \cdots, \alpha^{e-1}$ of the group where $e$ exceeds unity but does not exceed the order of the group element $\alpha$.

All four theorems are equivalent and to each may be appended a corollary in which the conclusions are sharpened. Thus the corollary to (I) is that $U$ has $n$ basis vectors $(x_i)=(x_{i1}, \cdots, x_{in})$ $(i=1, \cdots, n)$ which may be chosen in the form $x_{ii}=1$, $x_{ij}=0$ $(j>i)$. Theorem I is new and is due to the author; (Ia) is the famous Minkowski conjecture which was recently proved by Hajós [Math. Z. **47**, 427–467 (1941); these Rev. **3**, 302] who also proved a more general theorem on $k$-coverings of $n$-dimensional space. Hajós's proof is in three parts: (i) demonstration that (Ib) need be proved for "rational" lattices only; (ii) proof of equivalence of (Ib) and (Ic); (iii) proof of (Ic). By far the hardest part is (iii), and the author states that he is publishing a simplified proof elsewhere. The purpose of his present paper is to complete the picture by giving a simplified proof of the equivalence of (Ic) with the other three theorems. The steps in the argument are as follows: (a) (Ia) and (Ib) are equivalent. (b) (I) and (Ib) are equivalent. (c) If (Ib) is true for rational lattices it is true in general. (A very simple proof of this is given.) (d) There is a correspondence between finite modules and rational lattices. (e) (Ib) and (Ic) are equivalent.

*R. A. Rankin* (Cambridge, England).

**Kuo, Huan-Ting. A recurrence formula for $\zeta(2n)$.** Bull. Amer. Math. Soc. **55**, 573–574 (1949).

The author derives a recurrence relation for $\zeta(4n)$ which requires only the values $\zeta(2), \zeta(4), \cdots, \zeta(2n)$, at the expense of being quadratic rather than linear. The idea is to use the identity $(\pi-x)/2 = \sum_{k=1}^{\infty} k^{-1} \sin kx$, integrate $2n-1$ times between 0 and $x$ and then apply Parseval's formula to the resulting equation.

*R. Bellman.*

**Čudakov, N. G. On some power series containing prime numbers as exponents.** Doklady Akad. Nauk SSSR (N.S.) **65**, 445–448 (1949). (Russian)

The author investigates the behavior of the function $f(x)=\sum_{p \geqq 2} \log p \cdot x^{p^k}$ in the neighborhood of the rational points of the unit circle; the summation letter $p$ runs over all primes. For $k=1$, the author treated this previously [Ann. of Math. (2) **48**, 515–545 (1947); these Rev. **9**, 11]. He now extends his method to the case $k>1$. This result may be stated as follows. Let $x=r \exp i\varphi$, $0 \leqq r < 1$, $M=(1-r)^{-1}$, $(a, q)=1$, $\rho=\exp 2\pi i a/q$, $\vartheta=\varphi-2\pi a/q$, $A$ be a fixed positive constant, $\tau_1=(\log M)^{A_1}$ where $A_1$ (as well as $A_2$ mentioned later) is a certain function of $k$ and $A$, and let $\psi_\rho(x)=\{\varphi(k)(1-x/\rho)^{1/k}\}^{-1}\Gamma(1/k)\sum'_n \rho^{n^k}$, where the prime indicates that $n$ runs through a reduced residue system

mod $q$. Then on the Farey arc of order $\tau_1$, $|\vartheta| \leqq \pi/(q\tau_1)$, $q \leqq \tau_1$, we have $f(x)-\psi_\rho(x) = M^{1/k}(\log M)^{1+1/k-A}O(1)$. If $|\vartheta| \leqq 2\pi(\log M)^{A_3}/M$, the treatment is the same as in the paper mentioned above. However, when

$$|\vartheta| > 2\pi(\log M)^{A_3}/M,$$

corresponding to the minor arc in the usual treatment, the author finds it necessary to appeal to known estimates for the exponential sum $\sum_{p \leqq x} p^k \exp 2\pi i a/q$; in his previous work, exponential sums were avoided.

The author remarks that his theorem implies the known Goldbach-Waring theorem [see L. K. Hua, The additive prime number theory, Trav. Inst. Math. Stekloff 22 (1947); these Rev. 10, 597] on the number of representations of an integer as the sum of $s$ $k$th powers of primes. He promises further applications later. *L. Schoenfeld.*

**Nečaev, V. I. The representation of integers by sums of terms of the form $\dfrac{x(x+1)\cdots(x+n-1)}{n!}$.** Doklady Akad. Nauk SSSR (N.S.) **64**, 159–162 (1949). (Russian)

Let $n$ be an integer not less than 2,

$$f(x) = a_n x^n + \cdots + a_1 x, \qquad a_n > 0,$$

where the $a_i$ are integers, $(a_1, \cdots, a_n) = 1$; and denote by $d$ the greatest common divisor of the integers $f(x)$ for all integral $x$. The author defines $G(f)$ to be the least integer $r$ for which there exists an integer $c$ such that all integral $N \geqq c$ can be represented in the form

$$N = f(x_1)/d + \cdots + f(x_r)/d, \tag{1}$$

with $x_1, \cdots, x_r$ nonnegative integers. If there exists a positive integer $x_0$ for which $f(x_0) = d$ then all integers $N \geqq 1$ can be represented in the form (1) for suitable $r$. In this case the author defines $g(f)$ to be the least $r$ such that all integers can be represented in the form (1) with $r = g(f)$. The case $f(x) = x^n$ is then that of the classical Waring problem. The author proposes to prove analogous results for these generalized $G(f)$, $g(f)$, using the method of Vinogradov; and includes a brief sketch of his proofs.

Let $d'$ denote the greatest common divisor of $f'(x)$ for all integral $x$; and for $p$ a prime dividing $d'$ define $\beta = \beta(p)$ to be such that there exists an integer $b$ having the property that for any integer $m$, and $s \geqq \beta$, the congruence $f(x)/d \equiv b + mp^\beta \pmod{p^s}$ has a solution. Then if $p_1, \cdots, p_l$ are the primes dividing $d'$, set

$$r_0 = \max\,(2n+1, p_1^{\beta_1}, \cdots, p_l^{\beta_l}).$$

The main theorem asserted is that

$$G(f) \leqq \begin{cases} \min\left(r_1, r_0 + r_2 - 2n\log\dfrac{r_0}{2n}\right) & \text{for } r_0 < r_1; \\ r_0 & \text{for } r_0 \geqq r_1, \end{cases}$$

where for $n \geqq 12$, $r_1 = [10n^2 \log n]$,

$$r_2 = 4n\log n + 2n\log\log n + 3.2n;$$

for $4 \leqq n \leqq 11$, $r_1 = 2^n + 1$, $r_2 = 2n(n-2)\log 2 + 4$.

In the special case $f(x) = \varphi(x) = x(x+1)\cdots(x+n-1)$, as well as in any other case where $d' = 1$, the author obtains $G(f) \leqq A(n)$, where $A(n)$ is given by the following table:

| $n=$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| $A(n)=$ | 5 | 8 | 17 | 31 | 45 | 63 | 81 | 103 | 125 | 155 |

and for $n \geqq 12$, $A(n) = 4n\log n + 2n\log\log n + 5.5n$. Also for this special $\varphi(x)$ the author gives the theorem that

$$n + \left[\frac{n}{2}\right] \leqq g(\varphi) \leqq \begin{cases} \frac{1}{2}n^2\log n + 8n\log n, & n \geqq 14; \\ n^2\log 2 + 8n, & 4 \leqq n \leqq 13. \end{cases}$$

*H. N. Shapiro* (New York, N. Y.).

**Fridlender, V. R. On the least $n$th-power non-residue.** Doklady Akad. Nauk SSSR (N.S.) **66**, 351–352 (1949). (Russian)

Let $n$ be a positive integer greater than 1 and for every prime number $p \equiv 1 \pmod n$ let $T_n(p)$ denote the smallest positive integer which is not congruent to an $n$th power modulo $p$. Using Linnik's theorem that the smallest prime in the arithmetic progression $kx+l$, $(k, l) = 1$, does not exceed $k^c$ for some positive constant $c$ [Rec. Math. [Mat. Sbornik] N.S. **15(57)**, 139–178, 347–368 (1944); these Rev. **6**, 260], the author proves that $T_2(p) = \Omega(\log p)$, i.e., $\limsup_{p\to\infty} \{T_2(p)/\log p\} > 0$. He remarks that this result cannot be extended to $T_n(p)$ for $n > 2$, since Linnik's theorem has not been carried over to cyclotomic fields. However, he points out that a theorem of Linnik on $T_2(p)$ [C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 119–120 (1942); these Rev. **4**, 189] can be generalized to $T_n(p)$ for any $n > 1$. *P. T. Bateman* (Princeton, N. J.).

**Erdös, P. On some applications of Brun's method.** Acta Univ. Szeged. Sect. Sci. Math. **13**, 57–63 (1949).

Let $P(k, l)$ denote the least prime in the arithmetic progression $kx+l$, $0 < l < k$, $(l, k) = 1$; let $\phi(k)$ denote the Euler function; let $c_1$, $c_3$ and $c_6$ be arbitrary positive constants. The author employs the V. Brun method to prove the following three theorems. (I) There exist a constant $c_2 = c_2(c_1)$ and infinitely many integers $k$ so that $P(k, l) > (1+c_1)\phi(k)\log k$ for more than $c_2\phi(k)$ values of $l$. (II) There exists a constant $c_4 = c_4(c_3)$ such that for $c_4\phi(k)$ values of $l$, $P(k, l) < c_3\phi(k)\log k$. (III) Let $n$ be sufficiently large. Then there exists a constant $c_5 = c_5(c_6)$ and a sequence of primes $p_k < p_{k+1} < \cdots < p_{k+r} < n$, $r = [c_6 \log n]$, so that $p_{k+i+1} - p_{k+i} > c_5$, $i = 0, 1, \cdots, r-1$. The proof uses results of Schnirelmann [Landau, Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1930**, 255–276], Page [Proc. London Math. Soc. (2) **39**, 116–141 (1935)] and the author [Proc. Cambridge Philos. Soc. **33**, 6–12 (1937)] on the distribution of primes. Theorem III is an improvement on a result obtained by Sierpiński [Colloquium Math. **1**, 193–194 (1948); these Rev. **10**, 431]. *A. L. Whiteman.*

**Hua, L. K., and Reiner, I. On the generators of the symplectic modular group.** Trans. Amer. Math. Soc. **65**, 415–426 (1949).

Every unimodular matrix of order $n \geqq 2$ with rational integral elements is here proved to be a product of the matrices $U_1$, $U_2$, $U_3$, and their inverses, where

$$xU_1 = (x_2, x_3, \cdots, x_n, x_1), \quad xU_2 = (x_1, x_1+x_2, x_3, \cdots, x_n),$$

and

$$xU_3 = (-x_1, x_2, \cdots, x_n).$$

This is accomplished by using a known result and showing that $U_4$, where $xU_4 = (x_2, x_1, x_3, \cdots, x_n)$, is a product of the $U_1$, $U_2$, $U_3$ and their inverses. With the aid of this theorem it is then proved that the symplectic modular group (factor group with respect to its centrum of the group of the skew-symmetric bilinear form $(x_1y_{n+1} - x_{n+1}y_1) + \cdots$ in $2n$ pairs of indeterminates) over the rational integers is generated by

exactly four independent generators when $n>1$ and by two when $n=1$. The four matrices are $\mathfrak{T}_0$, $\mathfrak{R}_1$, $\mathfrak{R}_2$ and $\mathfrak{S}_0$, where

$$(x, y)\mathfrak{T}_0=(x_1, \cdots, x_n, x_1+y_1, y_2, \cdots, y_n),$$

$(x, y)\mathfrak{R}_i=(xU_i, yU_i'^{-1})$ for $i=1, 2$, and

$$(x, y)\mathfrak{S}_0=(-y_1, x_2, x_3, \cdots, x_n, x_1, y_2, y_3, \cdots, y_n).$$

It is remarked that the method can be extended to any Euclidean ring and some details are given for an analogous group over the Gaussian integers. Two lemmas employed are as follows. (1) Let $m$ be a nonzero integer, and let $T$ be an $n$-rowed symmetric matrix at least one of whose elements is not divisible by $m$. There exists a symmetric matrix $S$ with integral elements such that

$$0<|\det(T-mS)|<|m|^n.$$

(2) Let $A$ and $B$ satisfy $AB'=BA'$ and let $\det A\neq 0$. There exists a symmetric matrix $S$ such that either $B-AS=0$ or $0<|\det(B-AS)|<|\det A|$. *W. Givens.*

## ANALYSIS

**Ballieu, Robert. Sur des extensions de l'inégalité triangulaire.** Simon Stevin **26**, 129–134 (1949).

Let $a, b, c$ be the sides of a triangle and let $\gamma$ be the angle opposite $c$. The author sets $k=c^n/(a^n+b^n)$ and makes an exhaustive study of bounds for $k$. Sets of upper and lower bounds for $k$ are given for $n<0$, $0<n\leqq 1$, $1<n\leqq 2$, $2<n$. Further, for $n>1$ sets of bounds are given depending on the range of $\gamma$. *A. W. Goodman* (Lexington, Ky.).

**Crijns, L. On generalizations of the triangle inequality.** Simon Stevin **26**, 135 (1949) (Dutch).

Let $a, b, c$ be the sides of a triangle and let $\gamma$ be the angle opposite $c$. The author gives a simple proof that if $\gamma\leqq\pi/2$ and $n\geqq 2$ then $a^n+b^n\geqq c^n(\frac{1}{2})^{n-1}$. *A. W. Goodman.*

**Jecklin, H. Versuch einer Systematik des mathematischen Mittelwertbegriffs.** Comment. Math. Helv. **22**, 260–270 (1949).

The author defines the mean value functions $m(x_1, \cdots, x_n)$ by the condition

$$\min(x_1, \cdots, x_n)\leqq m(x_1, \cdots, x_n)\leqq\max(x_1, \cdots, x_n)$$

[in the usual terminology these are called internal means; cf., e.g., B. de Finetti, Giorn. Ist. Ital. Attuari **2**, 369–396 (1931), in particular, pp. 388–391] while he calls "$M$ Formeln" the functions satisfying $m(x, \cdots, x)=x$ for every $x$, which are usually called mean values. [The author's statement (p. 261, lines 26–33) that these internal means are all monotonic is erroneous as the example of the counter-harmonic mean $m=(x_1^3+x_2^3)/(x_1+x_2)$ shows; e.g., $x_2=10$, $x_1=3$, $x_1'=4$, $x_1''=6$.] The function $\phi\{[\sum F(X_i)g_i]/\sum g_i\}^{1/T}$ (where $X_i$ is a product of degree $T$ built from some $x_k-s$, $\sum X_i$ is symmetric in the $x_k-s$, $F(X)$ is monotonic and continuous in $X$ and has the inverse function $\phi$, and finally "$g_i$ is a positive function") which the author presents as the most general symmetric analytic mean value function [p. 266] will be symmetric only if further restrictions are laid upon the functions $g_i$ [counter-example: $(x_1\cdot x_1+x_2\cdot x_2^3)/(x_1+x_2^3)$] and it is not proved that all symmetric analytic means can be written in this form.

The author proves that the arithmetic and the geometric mean of two mean values (respectively, internal means) is again a mean value (respectively, an internal mean). He succeeds in constructing from an arbitrary function $\theta(x_1, \cdots, x_n)$ (or an arbitrary interval-function $\theta(\overline{x_1x_n})$) an internal mean $m=(x_1+x_ne^\theta)/(1+e^\theta)$ and shows that every internal mean has this form, as for an arbitrary internal $m$: $\theta=\log[(m-x_1)/(x_n-m)]$. *J. Aczél* (Szeged).

**Aczél, Jean. Sur les opérations définies pour nombres réels.** Bull. Soc. Math. France **76**, 59–64 (1948).

For an open or semi-open interval $(a, b)$, the author calls a function of two variables an "operation defined in the interval $(a, b)$," and designates it by $z=xoy$, provided that for each $x$ and $y$ in $(a, b)$ the corresponding value $z$ lies in $(a, b)$. It is shown that there exists a strictly increasing continuous function $f(x)$ in $(a, b)$, such that

$$xoy=f^{-1}[f(x)+f(y)], \tag{1}$$

if and only if $xoy$ is (i) strictly monotone: $xoy<x'oy$ for $x<x'$ (and similarly for $y<y'$); (ii) continuous:

$$\lim(xoy)=(\lim x)o(\lim y);$$

and (iii) associative: $(xoy)ow=xo(yow)$. Since the right-hand member of (1) is symmetric in $x$ and $y$, it follows that every strictly monotone, continuous, and associative operation defined in $(a, b)$ is also commutative. *E. F. Beckenbach* (Los Angeles, Calif.).

**Császár, Ákos. Sur les fonctions internes, non monotones.** Acta Univ. Szeged. Sect. Sci. Math. **13**, 48–50 (1949).

Call $f(x)$ a "fonction interne" on an open interval $(a, b)$ if, for all $x, y$ in the interval,

$$\min[f(x), f(y)]\leqq f(\tfrac{1}{2}(x+y))\leqq\max[f(x), f(y)],$$

equality holding only if $f(x)=f(y)$. Obvious examples are (i) $f(x)$ constant on $(a, b)$ and (ii) $f(x)$ any function strictly monotonic on $(a, b)$. The author shows that all other examples are non-Lebesgue-measurable on every measurable subset of $(a, b)$ of positive measure. More precisely, if $E$ is the set of $x$ in $(a, b)$ with $f(x)$ lying between $f(\alpha)$ and $f(\beta)$ for arbitrary (fixed) $\alpha, \beta$ in $(a, b)$, then the interior measure of $E$ is zero. This is a generalization of (part of) a result of A. Ostrowski [Jber. Deutsch. Math. Verein. **38**, 54–62 (1929)] for discontinuous solutions of $f(x+y)=f(x)+f(y)$. *I. Halperin* (Kingston, Ont.).

***Matos Peixoto, Mauricio. Convexity of curves.** Notas de Matemática, no. 6. Livraria Boffoni, Rio de Janeiro, 1948. i+66 pp. (Portuguese)

In the present book there is presented a brief discussion of the basic theory of convex functions as developed by Jensen and others, followed by a more extensive treatment of generalized convex functions. The latter development contains contributions which the author himself has made. The first of the four chapters is devoted to Jensen's theory of convex (sublinear) functions. It contains discussions of continuous and discontinuous convex functions, along with applications to inequalities and a suggestion of applications to complex variable theory.

In the second chapter the author develops known geometric aspects of sub-$\{F(x)\}$ functions, or functions convex relative to $\{F(x)\}$, where $\{F(x)\}$ denotes a family of functions continuous in an interval $a<x<b$ such that for each $(x_1, y_1)$, $(x_2, y_2)$ with $a<x_1<x_2<b$ there is a unique member of the family whose graph passes through the points

$(x_1, y_1)$, $(x_2, y_2)$. A sub-$\{F(x)\}$ function is not necessarily differentiable anywhere: the members of $\{F(x)\}$ are sub-$\{F(x)\}$ functions, and $\{F(x)\}$ might consist of the functions $\varphi(x)+mx+b$, where $\varphi(x)$ is nowhere differentiable. Nevertheless the author shows in the third chapter that under suitable restrictions on $\{F(x)\}$ every sub-$\{F(x)\}$ function is differentiable except at most on a set of measure zero; and that under certain further restrictions on $\{F(x)\}$ every sub-$\{F(x)\}$ function is differentiable except at most on a denumerable set, as in the case of ordinary convex functions.

Smooth convex functions are characterized by the differential inequality $y''\geqq 0$. In the final chapter the author establishes analogous differential inequalities for smooth functions convex relative to suitably restricted families $\{F(x)\}$; for a more complete discussion of this, see the following review. *E. F. Beckenbach* (Los Angeles, Calif.).

**Matos Peixoto, Mauricio. Generalized convex functions and second order differential inequalities.** Bull. Amer. Math. Soc. **55**, 563–572 (1949).

Let the family $\{F(x)\}$ of the book reviewed above be formed by the solutions of the differential equation $y''=G(x, y, y')$, where $G$ is continuous in $S$: $a<x<b$, $-\infty<y<+\infty$, $-\infty<y'<+\infty$, and such that for each $(x_0, y_0, y_0')$ in $S$ there is a unique solution $F(x)$ with $F(x_0)=y_0$, $F'(x_0)=y_0'$. The main result of the present paper is that under the above conditions a function $f(x)$ of class $C''$ is a sub-$\{F(x)\}$ function if and only if $f''\geqq G(x, f, f')$, $a<x<b$. A sufficient condition that $f(x)$ be a strictly sub-$\{F(x)\}$ function is $f''>G(x, f, f')$, $a<x<b$. An application is made to a theorem of Tchaplygin, Petrov, and Wilkins [Wilkins, Bull. Amer. Math. Soc. **53**, 126–129 (1947); these Rev. **8**, 464] concerning linear second order differential inequalities. *E. F. Beckenbach* (Los Angeles, Calif.).

**Obrechkoff, N. Sur quelques équations intégrales.** C. R. Acad. Bulgare Sci. Math. Nat. **1**, no. 2–3, 5–8 (1948).

Let $D$ be a fixed plane domain, and let $F(x, y)$ be a measurable function the integral of whose absolute value over the entire finite plane is convergent. The author investigates the problem of determining a function $f(x, y)$ of the same sort, for which $F(x, y)\equiv\iint_D f(x+\alpha, y+\beta)d\alpha d\beta$. The solution is given in terms of Laplace transforms. For $F(x, y)\equiv 0$ the problem reduces to one posed by Pompeiu; it is shown in this case that $f(x, y)$ must vanish at each point of continuity, so that the only continuous solution is $f(x, y)\equiv 0$. A similar problem involving contour integration is discussed, and a bibliography of papers concerning similar problems is given. *E. F. Beckenbach.*

**Germansky, Baruch. On function systems of Tchebyscheff belonging to a given interval.** Riveon Lematematika **3**, 33–36, 37 (1949). (Hebrew. English summary)

An infinite sequence of real functions $f_k(x)$ bounded and continuous on an open interval $I$ is a Tchebyscheff system if every "polynomial" $P(x)=a_0f_0(x)+\cdots+a_nf_n(x)$ of degree $n$ in the $f_k$ has at least $n$ zeros in $I$. The $a_k$ are arbitrary real numbers. If the zeros are counted once or twice according as $P(x)$ does or does not change sign, the author speaks of a T. system in the sense of Bernstein; if all zeros are counted once, he speaks of a T. system in the sense of Fekete. Clearly every T.-B. system is also a T.-F. system: the author's principal result is the converse, that every T.-F. system is also a T.-B. system. The proof is constructive. *A. Erdélyi* (Pasadena, Calif.).

**Petrovitch, Michel. Un mode général de représentation paramétrique des transcendantes d'ordre fini.** Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. **7**, 43–54 (1941).

[In the original "paramétrique" appeared as "paramérrique."] The following statement is made without proof. Let $y(x)$ satisfy an algebraic differential equation. Then $x$ and $y$ can be expressed parametrically as exponentials of sums of constant multiples of integrals of functions $u_i(t)$, $i=1, \cdots, p$, where $p$ is some integer. Each $du_i/dt$ is a sum of terms $a_{ij}e^{u_j}$, $j=1, \cdots, p$, where each $a$ is either 0 or 1. *J. F. Ritt* (New York, N. Y.).

**Mulè, Giovanni. Sulla dipendenza lineare delle funzioni di più variabili.** Pont. Acad. Sci. Acta **11**, 219–225 (1947).

The author proves the following conditions sufficient for the linear dependence of $m$ functions of $r$ variables. Given functions $u_i(x_1, x_2, \cdots, x_r)$, $i=1, 2, \cdots, m$, continuous together with their partial derivatives of all orders through $m-1$ in a region $C$, suppose that in $C$ the (Wronskian) determinant $W$ of the $u$'s and their partial derivatives with respect to $x_1$ through order $m-1$ is identically zero, as is also each of the $r-1$ determinants obtained by replacing the last row of $W$ by $\partial u_1/\partial x_k \cdots \partial u_m/\partial x_k$ $(k=2, 3, \cdots, m)$. If in addition the matrix of the first $m-1$ rows of $W$ has rank $m-1$ at all points of $C$, then $u_1, u_2, \cdots, u_m$ are linearly dependent in $C$.

If $W_i$ is the cofactor of the $i$th term of the last row of $W$, $R=(W_1^2+\cdots+W_m^2)^{\frac{1}{2}}$, and $\alpha_i=W_i/R$, then the $\alpha_i$ are constants and the linear relation is $\alpha_1u_1+\cdots+\alpha_mu_m\equiv 0$. The proof involves partial differentiation and inspection of determinants of coefficients. An application is made to vector analysis. A few obvious typographical errors appear. *A. B. Brown* (Flushing, N. Y.).

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## *Theory of Sets, Theory of Functions of Real Variables*

***Tarski, Alfred. Cardinal Algebras. With an Appendix: Cardinal Products of Isomorphism Types, by Bjarni Jónsson and Alfred Tarski.** Oxford University Press, New York, N. Y., 1949. xii+326 pp. $10.00.

This book is an axiomatic investigation of the novel types of algebraic systems which arise from three sources: the arithmetic of cardinal numbers; the formal properties of the direct product decompositions of algebraic systems; the algebraic aspects of invariant measures, regarded as functions on a field of sets.

The essential feature of the algebraic systems considered is the presence of "refinement" postulates. A cardinal algebra (C.A.) is a system $A$ with binary and denumerably infinite addition, subject to the postulates: (I) and (II) closure; (III) (associative law) $\sum a_i=a_0+\sum a_{i+1}$; (IV) (commutative-associative law) $\sum(a_i+b_i)=\sum a_i+\sum b_i$; (V) zero; (VI) (refinement) $a+b=\sum c_i$ implies $c_i=a_i+b_i$, $a=\sum a_i$, $b=\sum b_i$ for suitable $a_i$, $b_i$; (VII) (remainder) $a_n=b_n+a_{n+1}$ implies $a_n=c+\sum b_{n+i}$ for some $c$. Essentially, the same postulates, with closure omitted, define a generalized cardinal algebra (G.C.A.). A refinement algebra (R.A.) is a system with a binary, associative, and commutative addition which need not be closed under addition, which has a zero with $a+b=0$ implies $a=b=0$, and which satisfies two refinement postulates: (i) if $a_1+a_2=b_1+b_2$, then there are

$c_{ij}$ with $b_j=\sum_i c_{ij}$, $a_i=\sum_j c_{ij}$; (ii) if $a_1+a_2+c=b+c$, then there are $b_1, b_2, c_1, c_2$ with $b=b_1+b_2$, $c=c_1+c_2$, $a_i+c_i=b_i+c_i$. In any such algebra, we define $a\leqq b$ to mean that $b=a+c$ for some $c$.

The family of all sets under set-theoretic addition, and the family of all cardinal numbers under addition constitute C.A.'s. Almost all the additive properties of cardinal numbers which involve at most a denumerable use of the axiom of choice can be derived from the axioms for a C.A. The first part of the book, which is devoted to such a derivation, is, in effect, an elegant presentation of the arithmetic of cardinal numbers. Included are new proofs of the fundamental law of infinite addition ($a_0+\cdots+a_n\leqq b$ for every $n$ implies $\sum a_i\leqq b$) and of the cancellation law, $m\cdot a+c\leqq m\cdot b+c$ implies $a+c\leqq b+c$. Operations of meet and join (not always defined) are introduced in any C.A. in terms of the partial order $a\leqq b$ defined above, and the general distributive laws connecting these operations and addition are formulated. The idem-multiple ($a+a=a$), finite ($a+x=a$ implies $x=0$), and multiple free ($x+x\leqq a$ implies $x=0$) elements $a$ of a C.A. are studied.

The second part treats construction methods for cardinal algebras, such as the Cartesian product $\prod A_\alpha$, and more especially its subalgebra, the cardinal product, consisting of all functions $f$ with $f(\alpha)\in A_\alpha$ and $f(\alpha)\neq 0$ for at most a denumerable number of indices $\alpha$. For an equivalence relation $R$ on the elements of a cardinal algebra $A$, one may define the coset C.A. $A/R$, provided $R$ satisfies suitable additivity conditions designed to preserve the refinement postulates. These conditions, however, are so strong that not every homomorphic image of $A$ has the form of such a coset algebra $A/R$. An ideal $B$ in a C.A. is a subset closed under denumerable addition and such that $a\leqq b\in B$ implies $a\in B$. Congruence modulo an ideal yields a coset algebra, but not every coset algebra can be so obtained. The algebra of all ideals of a C.A., combined under union, is itself a C.A. in which every element is idem-multiple.

A cardinal algebra $A$ may also be regarded as a generalized cardinal algebra under either the operation $a\bigcup b$ (finite and denumerably infinite) or the operation of disjunctive addition, where $a+b=a+b$ is defined only if $a\bigcap b=0$, and similarly for infinite sums. This "disjunctive" algebra $A^*$ occurs naturally in the construction of the algebra of cardinal numbers from the algebra of sets; both constructions necessitate the use of generalized cardinal algebras (without closure). It is demonstrated that every G.C.A. can be embedded in an essentially unique C.A.; this makes it possible to apply results obtained for C.A.'s to G.C.A.'s.

The measure-theoretic studies concern a field $F$ of sets (regarded as a C.A.) together with a set $G$ of one-to-one transformations. If a transformation $f\in G$ is defined for all elements of a set $A\in F$, then $f(A)$ must be a set of $F$, $G$ must contain the identity and must be closed under products and inverses of transformations. Following the author [Fund. Math. 31, 47–66 (1938)] and generalizing results of von Neumann [Fund. Math. 13, 73–116 (1929)], one has the following theorems. If $U$ is a set in $F$, then there exists a finitely additive real-valued measure $m$, invariant under $G$, defined for all sets of $F$ and with $m(U)=1$, provided either (i) $G$ is commutative, or (ii) $F$ is the field of all subsets of a fixed set, and the set $U$ has no paradoxical decomposition (of the type appearing in the Banach-Tarski paradox). The proofs depend on the algebraic properties of the coset algebra $F^*/R$, where $F^*$ is the disjunctive algebra of $F$, and $R$ is the equivalence relation which holds between two sets $A$ and $B$ if $A$ and $B$ can be represented as finite disjoint sums $\sum^* A_i$, $\sum^* B_i$ with each $A_i$ congruent to $B_i$ under a transformation of the group $G$. Similar constructions for arbitrary cardinal algebras are also carried out [in § 11]; the reader who finds these constructions obscure is advised to consult this application first.

The third part of the book treats connections with other algebraic systems. In an algebraic system $A$ with a binary operation of addition, there is at most one way of introducing denumerably infinite addition so as to obtain a C.A. or a G.C.A.; when this is possible, $A$ itself is called a C.A. or a G.C.A. Conditions that a semi-group be a C.A. or an R.A. are obtained; there are close relations with the theory of partially ordered groups. There are similar conditions that a lattice (regarded as an algebra under the operation of join) or a Boolean algebra be a C.A. Boolean algebras are described by a postulate system given in terms of the operation of disjunctive addition. The cardinal algebras of integers and of real numbers (with $\infty$ adjoined) are characterized; the algebra of relation numbers is shown to have a suitable unique decomposition theorem.

The isomorphism types of any class of algebras may themselves be regarded as an algebra under the finite or denumerably infinite operations of forming Cartesian products. The algebra of isomorphism types of all cardinal algebras (all finitely closed G.C.A.'s) is a C.A. (an R.A.). The algebra of isomorphism types of all countably distributive (and countably complete) lattices is an R.A. (respectively, a C.A.). The essential content of these results is the validity of refinement theorems for the Cartesian product operation.

The appendix considers algebras $A$ with a binary operation which has a zero; this hypothesis makes it possible to regard the Cartesian product as an "internal" product of subalgebras. The algebra $\mathfrak{F}(A)$ of all subalgebras which are direct factors of $A$, and the algebra $\phi(A)$ of all isomorphism types of (Cartesian) factors of $A$ are studied. If $A$ has a strong zero ($a+b=0$ implies $a=b=0$), then $\mathfrak{F}(A)$ is a disjunctive Boolean algebra and satisfies the refinement property (i) for an R.A.; all algebras of this type have a unique decomposition into indecomposable direct factors. Similar conditions that $\mathfrak{F}(A)$ be a countably complete disjunctive Boolean algebra are obtained. The binary addition considered in these results may be replaced by an arbitrary binary relation $R$ with a suitable zero, and the study of isomorphism types then becomes a study of certain relation numbers.

The book is replete with novel algebraic notions; it is written in logical style; all theorems (important and unimportant) are explicitly stated, and the proofs are carefully cross-referenced. *S. MacLane* (Chicago, Ill.).

**Tarski, Alfred. Axiomatic and algebraic aspects of two theorems on sums of cardinals.** Fund. Math. 35, 79–104 (1948).

The two theorems are as follows. ($C_1$) Given any infinite sequence of cardinals $\mathfrak{a}_0, \mathfrak{a}_1, \cdots, \mathfrak{a}_n, \cdots$ and a cardinal $\mathfrak{b}$, if $\mathfrak{b}<\sum_{n<\omega}\mathfrak{a}_n$, then there is a natural number $p$ such that $\mathfrak{b}\leqq\sum_{n<p}\mathfrak{a}_n$. ($C_2$) Given any infinite sequence of cardinals $\mathfrak{a}_0, \mathfrak{a}_1, \cdots, \mathfrak{a}_n, \cdots$ and a cardinal $\mathfrak{b}$, if $\sum_{n<p}\mathfrak{a}_n\leqq\mathfrak{b}$ for every natural number $p$, then $\sum_{n<\omega}\mathfrak{a}_n\leqq\mathfrak{b}$. Although both theorems express the fact that the sum of an infinite sequence of cardinals is the least upper bound of the partial sums, there is a remarkable difference between them, for $C_1$ is equivalent to the axiom of choice, while the proof of $C_2$ uses only

the restricted axiom of choice (the axiom of choice applied to countable families of sets).

Both theorems can be stated in terms of sets and set-theoretical equivalence; this suggests a more abstract formulation in terms of a Boolean algebra and a binary relation between its elements. A binary relation $R$ between the elements of a countably complete Boolean algebra $\mathfrak{B}$ is said to be countably additive if, given any elements $\mathfrak{a}_0, \mathfrak{a}_1, \cdots, \mathfrak{a}_n, \cdots$ and $\mathfrak{b}_0, \mathfrak{b}_1, \cdots, \mathfrak{b}_n, \cdots$ in $\mathfrak{B}$ such that $\mathfrak{a}_n \cdot \mathfrak{a}_p = 0 = \mathfrak{b}_n \cdot \mathfrak{b}_p$ for $n < p < \infty$ and $\mathfrak{a}_n R \mathfrak{b}_n$ for $n < \infty$, then $\sum_{n<\infty} \mathfrak{a}_n R \sum_{n<\infty} \mathfrak{b}_n$. The relation $R$ is called (finitely) refining if, given any elements $\mathfrak{a}$, $\mathfrak{a}_1$, $\mathfrak{a}_2$, and $\mathfrak{b}$ in $\mathfrak{B}$ such that $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$, $\mathfrak{a}_1 \cdot \mathfrak{a}_2 = 0$, and $\mathfrak{a} R \mathfrak{b}$, then there are elements $\mathfrak{b}_1$ and $\mathfrak{b}_2$ for which $\mathfrak{b} = \mathfrak{b}_1 + \mathfrak{b}_2$, $\mathfrak{b}_1 \cdot \mathfrak{b}_2 = 0$, $\mathfrak{a}_1 R \mathfrak{b}_1$, and $\mathfrak{a}_2 R \mathfrak{b}_2$. Several examples of countably additive and refining relations are given. The following analogue of $C_2$ holds. ($Q_2$) Let $\mathfrak{a}_0, \mathfrak{a}_1, \cdots, \mathfrak{a}_n, \cdots$ and $\mathfrak{b}$ be elements of a countably complete Boolean algebra $\mathfrak{B}$, and let $R$ be a countably additive and refining relation between elements of $\mathfrak{B}$. If, for every natural number $p$, there is an element $\mathfrak{c}$ in $\mathfrak{B}$ such that $\sum_{n<p} \mathfrak{a}_n R \mathfrak{c}$ and $\mathfrak{c} \leqq \mathfrak{b}$, then there is an element $\mathfrak{d}$ in $\mathfrak{B}$ for which $\sum_{n<\infty} \mathfrak{a}_n \leqq \mathfrak{d}$ and $\mathfrak{d} \leqq \mathfrak{b}$. It appears to be impossible to obtain a similar generalization of $C_1$.

Although the proof of $Q_2$ is based on the same ideas as the proof of $C_2$, it uses a different form of the axiom of choice, namely the principle of an infinite sequence of successive choices: if $A$ is a nonempty set and $S$ a relation such that for every element $x \varepsilon A$ there is an element $y \varepsilon A$ with $xSy$, then there exists an infinite sequence of elements $x_0, x_1, \cdots, x_n, \cdots$ in $A$ such that $x_n S x_{n+1}$ for $n = 0, 1, 2, \cdots$. The author raises the question of whether this principle is equivalent to the restricted axiom of choice.

Several other theorems on cardinals are mentioned which can be proved without using the most general form of the axiom of choice and which, like $C_2$, can be given in a purely algebraic form. Among these are the Cantor-Bernstein theorem and the following interpolation theorem. Given any two sequences of cardinals $\mathfrak{a}_0, \mathfrak{a}_1, \cdots, \mathfrak{a}_n, \cdots$ and $\mathfrak{b}_0, \mathfrak{b}_1, \cdots, \mathfrak{b}_n, \cdots$, if $\mathfrak{a}_n \leqq \mathfrak{b}_p$ for $n, p = 0, 1, 2, \cdots$, then there is a cardinal $\mathfrak{c}$ such that $\mathfrak{a}_n \leqq \mathfrak{c} \leqq \mathfrak{b}_p$ for $n, p = 0, 1, 2, \cdots$. The algebraic aspects of these theorems are discussed in a greater detail in the author's book [see the preceding review]. *B. Jónsson* (Providence, R. I.).

**Mostowski, Andrzej. On the principle of dependent choices.** Fund. Math. **35**, 127–130 (1948).

The paper is concerned with systems of set-theory that are consistent, and remain so after the adjunction of the axiom ($N$): there is a noncountable set of elements that are not sets. The system $\mathfrak{S}$ described in the author's paper (1) [Fund. Math. **32**, 201–252 (1939)] is such a system, provided that $\mathfrak{S}$ itself is consistent. The main result is that in such a system the general axiom of choice is independent of the following "weak axiom of choice" ($T$): if $R$ is a binary relation and $B$ a set $\neq 0$ and if for every $x \varepsilon B$ there is a $y \varepsilon B$ such that $xRy$, then there is a sequence $x_1, x_2, \cdots, x_n, \cdots$ of elements of $B$ such that $x_n R x_{n+1}$ for $n = 1, 2, \cdots$ [cf. the preceding review]. A model is made in $\mathfrak{S}_1 = \mathfrak{S} + (N)$ resembling one used in the paper (1), in which all the axioms of $\mathfrak{S}$, and ($T$), correspond to provable propositions in $\mathfrak{S}_1$; but the axiom of choice corresponds to a false proposition.

If $\mathfrak{m}$ is a cardinal number definable in Bernays' set-theory [J. Symbolic Logic **2**, 65–77 (1937); **6**, 1–17 (1941); these Rev. **2**, 210]; if $Z^*(\mathfrak{m})$ is the axiom: "if $A$ is a set with cardinal number less than $\mathfrak{m}$ and if every element of $A$ is a nonvoid set, then there is a function $f$ such that $f(X) \varepsilon X$ for $X \varepsilon A$"; and if $Z(\mathfrak{m}, 2)$ is obtained from $Z^*(\mathfrak{m})$ by replacing "nonvoid set" by "nonvoid set with cardinal not exceeding 2"; then it can be shewn by similar methods [the author says] that $Z^*(\mathfrak{m}) \rightarrow Z(\mathfrak{m}, 2)$ is not provable in $\mathfrak{S}$ (supposed consistent).

*M. H. A. Newman* (Manchester).

**Rado, R. Covering theorems for ordered sets.** Proc. London Math. Soc. (2) **50**, 509–535 (1949).

The author's introduction is as follows. Let $S$ be an abstract set. The elements of $S$ will be called points. $S$ is supposed to be ordered by means of a relation $a < b$. This means that, given any points $a$ and $b$, exactly one of the three relations $a < b$, $a = b$, $b < a$ holds, and the relations $a < b$, $b < c$ imply $a < c$. A subset $I$ of $S$ is called an interval of $S$ if it satisfies the following condition. Whenever $a < c < b$, $a, b \varepsilon I$ then $c \varepsilon I$. There need not exist any points which play the roles of end-points of $I$, even in the case of "bounded" intervals. For let $S$ be the set of all rational numbers, ordered according to magnitude, and let $I$ consist of all rational numbers $x$ satisfying $2 < x^3 < 3$. This fact introduces a certain amount of complication into our arguments.

Let $N$ be an abstract set. The elements of $N$ will be called indices. Suppose that with every index $\nu$ is associated an interval $I_\nu$ of $S$. The possibility is not excluded that there are pairs of distinct indices $\nu, \nu'$ satisfying $I_\nu = I_{\nu'}$. In theorem 1 of this note the following problem is solved. Under what circumstances is it possible to divide the system of intervals $I_\nu$, i.e., the index set $N$, into a given number $k$ of parts in such a way that no two intervals $I_\nu$ belonging to the same part have a point in common? An obviously necessary condition is that no $k+1$ intervals $I_\nu$ belonging to $k+1$ distinct indices $\nu$ should have a point in common. Theorem 1 asserts that this condition is also sufficient. It will be shown that this result is equivalent to the following proposition: If any system of $k+1$ distinct indices $\nu_1, \nu_2, \cdots, \nu_{k+1}$ can be subdivided into $k$ groups in such a way that no two intervals $I_{\nu_s}$ belonging to indices of the same group have a point in common, then the same kind of subdivision is possible for the whole system $N$ of indices.

Theorem 2 deals with the following problem. Under what circumstances is it possible to find a finite number of subsystems of a given system of intervals $I_\nu$ which have the property that no two intervals belonging to the same subsystem have a point in common and, at the same time, the set of points covered by the intervals of the subsystems coincides with the set of points covered by all intervals of the given system $I_\nu$? A necessary and sufficient condition will be shown to be the following. It should be possible to define a well-ordering of the index set $N$ which is such that the set $N(x)$ of all indices $\nu$ whose corresponding intervals $I_\nu$ contain any given point $x$ possesses a last element. Furthermore, it will be shown that, if any finite number $k$ of subsystems of the given system $I_\nu$ can be found which possess the above-mentioned property, then this is always possible with $k = 2$. The main idea of the proof of theorem 2 is closely connected with a method used by Denjoy for proving a more special covering theorem.

Theorems 3 and 4 are covering theorems for the set of all real numbers. Theorem 3 is an extension of the one-dimensional case of Lindelöf's covering theorem. Theorems 4 and 5 are quantitative versions of theorem 2 in which a "small" set of uncovered points is admitted. *J. F. Randolph.*

**Rothberger, Fritz. On some problems of Hausdorff and of Sierpiński.** Fund. Math. **35**, 29–46 (1948).

The family $\mathfrak{N}$ of all sets of natural numbers is partially ordered under the nonreflexive order relation $A<B$ meaning that the set $B-A$ is infinite and the set $A-B$ is finite. A transfinite sequence of $\mathfrak{N}$-sets $A_\mu$ ($\mu<\Omega$) with $A_\mu<A_\nu$ for all $\mu<\nu\leqq\Omega$ is said to be an $\Omega$ limit if for no set $X$ is $A_\mu<X<A_\Omega$ for all $\mu<\Omega$. This paper concerns the existence of $\Omega$ limits. It is shown that if no $\Omega$ limits exist, then every subset of a linear set of power $\aleph_1$ is a relative $\mathfrak{F}_\sigma$, and the family of all real functions defined on a set of power $\aleph_1$ has a denumerable base in the sense of function limits; these two conclusions being equivalent. It is then deduced that $\Omega$ limits exist if any one of the following propositions is true: $2^{\aleph_0}<2^{\aleph_1}$; a Lebesgue nonmeasurable linear set of power $\aleph_1$ exists; a second category linear set of power $\aleph_1$ exists. Furthermore, $\Omega$ limits exist if $\Omega, \omega$ gaps exist; such a gap being a pair of transfinite sequences of $\mathfrak{N}$-sets $A_\mu$ ($\mu<\Omega$) and $B_m$ ($m<\omega$) with $A_\mu<A_\nu<B_n<B_m$ for all $\mu<\nu<\Omega$, $m<n<\omega$ such that for no $\mathfrak{N}$-set $X$ is $A_\mu<X<B_m$ for all $\mu<\Omega$, $m<\omega$. It is known that $\Omega, \Omega$ gaps exist and that $\omega, \omega$ gaps do not exist. *W. Gustin* (Bloomington, Ind.).

**Sierpiński, Wacław. Sur les ensembles presque contenus les uns dans les autres.** Fund. Math. **35**, 141–150 (1948).

In extension of the terminology used in the above review let $A\leqq B$ mean that the set $A-B$ is finite, and let $A=B$ mean that $A\leqq B$ and $B\leqq A$. Define a pair of families of sets $\mathfrak{A}$ and $\mathfrak{B}$, the family $\mathfrak{A}$ having power $\mathfrak{a}$ and the family $\mathfrak{B}$ having power $\mathfrak{b}$, to be an $\mathfrak{a}, \mathfrak{b}$ (cardinal) gap if $A\leqq B$ for all $A\in\mathfrak{A}$, $B\in\mathfrak{B}$ and if for no set $X$ is $A\leqq X\leqq B$ for all $A\in\mathfrak{A}$, $B\in\mathfrak{B}$. It is known that no $\aleph_0, \aleph_0$ gaps exist and that $\aleph_1, \aleph_1$ gaps do exist. It is shown that $2^{\aleph_0}, \aleph_0$ gaps in $\mathfrak{N}$ exist, so that under the continuum hypothesis $\aleph_1, \aleph_0$ gaps in $\mathfrak{N}$ exist. It is also shown with the aid of the continuum hypothesis that $\Omega, \omega$ (ordinal) gaps and $\Omega$ limits exist. Finally, again using the continuum hypothesis, an $\mathfrak{N}$-set $Y$ and a pair of transfinite sequences of $\mathfrak{N}$-sets $A_\mu$ ($\mu<\Omega$) and $B_\mu$ ($\mu<\Omega$) are constructed with $A_\mu<A_\nu<Y<B_\nu<B_\mu$ for all $\mu<\nu<\Omega$ such that $X=Y$ for any set $X$ having the property that $A_\mu<X<B_\mu$ for all $\mu<\Omega$. *W. Gustin* (Bloomington, Ind.).

**Sierpiński, Wacław. Exemple effectif d'une famille de $2^{\aleph_1}$ ensembles linéaires croissants.** Fund. Math. **35**, 213–216 (1948).

The object of this note is to define effectively a family $F$ of $c$ ($=2^{\aleph_0}$) linear, increasing sets (i.e., such that each of these sets is a sub- or superset of each of the others). The chief aid is the following lemma, whose proof is a modification of an idea of Lebesgue [J. Math. Pures Appl. (6) **1**, 139–216 (1905), in particular, p. 213]. One can effectively define a function $f(D)$ which mates with each denumerable set $D$ of ordinals greater than $\omega$ and less than $\Omega$ a linear, nonnull set $f(D)$ such that $f(D)$ and $f(D')$ are disjoint whenever $D$ and $D'$ are distinct. The proof proceeds by considering the infinite sequence $r_1, r_2, \cdots$ of all rationals, and representing an irrational $x$, $0<x<1$, by the continued fraction

$$x=\cfrac{1}{\nu(1,x)+\cfrac{1}{\nu(2,x)+\cdots}},$$

where the $\nu$'s are uniquely determined positive integers. If $u_1, u_2, \cdots$ is an infinite sequence of distinct rationals, $\varphi(u_1, u_2, \cdots)$ denotes the order type of the $u_n$ when arranged according to magnitude. If, now, $D$ is a denumerable set of ordinals greater than $\omega$ and less than $\Omega$, $f(D)$ is defined as the set of all the irrationals $x$, $0<x<1$, such that the set of order types

$$\varphi(r_{\nu(2^k-1,x)}, r_{\nu(3\cdot 2^{k-1},x)}, r_{\nu(5\cdot 2^{k-1},x)}, \cdots)$$

with $k=1, 2, \cdots$, is identical with $D$ except permissibly for the order of its terms. It follows at once from the result of this note, by the adjunction of the continuum hypothesis [as was proved in a different manner by the author in his Hypothèse du Continu, Warsaw, 1934, p. 120] that the continuum is the sum of $c$ increasing sets.

*H. Blumberg* (Columbus, Ohio).

**Sierpiński, Waclaw. Sur une décomposition de la droite.** Comment. Math. Helv. **22**, 317–320 (1949).

Proof, by means of the continuum hypothesis, of the following theorem. There exists a decomposition of the straight line $X$ into $c$ ($=2^{\aleph_0}$) disjoint sets, each of cardinal $c$, such that every translation along $X$ transforms each of the sets into itself, a countable (finite or denumerable) exceptional set being allowable. The desired components are defined as follows. By the continuum hypothesis $X$ may be written as a transfinite sequence of type $\Omega$

$$(1)\qquad x_1, x_2, \cdots, x_\omega, \cdots, x_\xi, \cdots, \qquad \xi<\Omega.$$

A transfinite sequence $p_\alpha$, $\alpha<\Omega$, is defined inductively as follows. Let $p_1=x_1$, and suppose, for $1<\alpha<\Omega$, that $p_\xi$ has been defined for $\xi<\alpha$. Consider the set $P_\alpha$ of all the numbers $p_\xi\pm x_{\xi_1}\pm x_{\xi_2}\pm\cdots\pm x_{\xi_n}$, where $\xi<\alpha, \xi_i<\alpha$ for $i=1, \cdots, n$, and $n$ is any positive integer. Define $p_\alpha$ as the first term of (1) in $X-P_\alpha$ (which is not empty). The set $Z$ of ordinals less than $\Omega$ may be decomposed into $\aleph_1$ disjoint sets $Z_\lambda$, $1\leqq\lambda<\Omega$, each of cardinal $\aleph_1$. For $\lambda<\Omega$, let $E_\lambda$ be the set of all numbers $p_\alpha\pm x_{\xi_1}\pm x_{\xi_2}\pm\cdots\pm x_{\xi_n}$, with $\xi_i<\alpha$, $i=1, \cdots, n$, and $\alpha\in Z_\lambda$. It is these $E_\lambda$ which are the components of $X$ for which the property of the theorem is proved. It may be shown without the continuum hypothesis that there exists a decomposition of $X$ into $c$ disjoint sets, each of cardinal $c$, such that every translation along $X$ transforms each of the sets into itself, an exceptional set of cardinal less than $c$ being allowable. *H. Blumberg* (Columbus, Ohio).

**Levi, Beppo. Sopra l'aritmetica transfinita.** Boll. Un. Mat. Ital. (3) **4**, 1–6 (1949).

Expository article.

***Eyraud, Henri. Leçons sur la Théorie des Ensembles, les Nombres Transfinis et le Problème du Continu. II.** Institut de Mathématiques, Lyon, 1947. 28 pp. [paged 66–93].

A continuation of notes of a course of lectures [see these Rev. **9**, 230]. This part is mainly concerned with ideas of the author's earlier paper [C. R. Acad. Sci. Paris **224**, 169–171 (1947); these Rev. **8**, 320] on asymptotic divisibility.

*J. Todd* (Washington, D. C.).

**Eyraud, Henri. Les tronçons du réseau $R(F)$.** Ann. Univ. Lyon. Sect. A. (3) **10**, 39–42 (1947).

**Eyraud, Henri. Le théorème du continu.** Ann. Univ. Lyon. Sect. A. (3) **11**, 5–8 (1948).

Cf. the preceding review.

**Kestelman, H., and Smith, C. A. B. On the distances between the elements of a subset of a group.** J. London Math. Soc. **24**, 131–135 (1949).

The classical nonmeasurable subset $S$ of real numbers consists of exactly one number from each coset of the subgroup of the rationals; the set of all reals is then the set-union of a countable number of disjoint, superposable subsets of the form $S+c$ ($c$ runs over all rationals) whose common set of distances $x-y$ ($x, y$ in $S$) exclude arbitrarily small numbers. A similar decomposition of all real numbers into $k$ disjoint superposable subsets, whose set of distances exclude arbitrarily small numbers, is now given, for every integer $k>1$, using the following theorems. (1) If $p$ is any fixed prime not a factor of $k$ then the set $G_2$ of all numbers $kmp^n$ is a subgroup of index $k$ of the additive group $G_1$ of all numbers $mp^n$ ($m, n$ arbitrary integers, including negative ones). (2) Let $G$ denote an arbitrary (multiplicatively written) group and for any subset $S$ let $Q(S)$ denote the set of all quotients $x^{-1}y$ ($x, y$ in $S$). Suppose that $G_1$ is any subgroup of $G$ and $G_2$ is a normal subgroup of $G_1$ of index $k$ (finite or infinite) in $G_1$. Then $G$ is the set-union of $k$ disjoint sets of the form $E\rho$ ($E$ fixed, $\rho$ varying) such that the set-complements of $G_2$ with respect to $G_1$ and each of the $Q(E\rho)$ are disjoint. *I. Halperin* (Kingston, Ont.).

**Hayashida, Tsuyoshi. Note on a measure problem.** Kōdai Math. Sem. Rep., no. 1, 8 (1949).

If $\{A_i\}$ is a sequence of measurable subsets of a measure space of measure 1, then corresponding to every positive integer $N$ and every positive number $\epsilon$ there exists a positive integer $n_0$ such that, whenever $n \geqq n_0$, integers $i_1, \cdots, i_N$ can be found with $1 \leqq i_1 < \cdots < i_N \leqq n$, and with the property that $\mu(A_i \bigcap \cdots \bigcap A_i) > (n^{-1}\sum_{i=1}^{n}\mu(A_i))^N - \epsilon$.

*P. R. Halmos* (Chicago, Ill.).

**Dehn, Max. On the approximation of a function by a power series.** Math. Student **15** (1947), 79–82 (1949).

Let $f(x)$ be defined in the neighborhood of $x=a$, and let $n$ be a fixed positive integer. This note concerns the existence of constants $a_1, \cdots, a_n$ and of a function $\epsilon(h)$ such that $\lim_{h\to 0}\epsilon(h)=0$ and

$$\text{(I)} \qquad f(a+h)=f(a)+ha_1+h^2a_2+\cdots+h^n(a_n+\epsilon(h)).$$

It is known [see, e.g., P. Franklin, A Treatise on Advanced Calculus, Wiley, New York, 1940, p. 125 and exercise 19, p. 147; these Rev. **2**, 77] that the existence of such an approximation formula for $n>1$ does not imply the existence of higher order derivatives of $f(x)$. The author defines $R_1(h)=h^{-1}\{f(a+h)-f(a)\}$, $R_k(h)=h^{-1}\{R_{k-1}(2h)-R_{k-1}(h)\}$, and proves that an approximation formula (I) exists if and only if the limits $S_k=\lim_{h\to 0}R_k(h)$ exist when $k=1, 2, \cdots, n$. Also, in this case, $S_k=1\cdot 3\cdot \cdots \cdot(2^{k-1}-1)a_k$.

*A. E. Taylor* (Los Angeles, Calif.).

**Banerjee, D. P. On the zeros of a non-differentiable function.** Bull. Calcutta Math. Soc. **40**, 145–146 (1948).

The author states that "the zeros of $w(x)=\sum_{n=1}^{\infty}a^n \sin b^n x$ between $-\pi$ and $\pi$ are $\pm(\pi/b^k)(1+\lambda_k/2)$ where $k$ is any positive integer, $0<\lambda_k<1$, $a<1$, $b$ a positive integer of the form $4p+1$, $ab>1+3\pi/2$," but he merely proves that $w(\pi/b^k)>0$, $w(3\pi/2b^k)<0$ and hence that $w(x)$ has at least one zero between $\pi/b^k$ and $3\pi/2b^k$ (he omits to note that there must also be at least one zero between $3\pi/2b^k$ and $\pi/b^{k-1}$). He also proves some results which Srinivasienger obtained for the function $\sum a^n \cos b^n x$ [J. Indian Math. Soc. (N.S.) **3**, 114–117 (1938)]. *F. A. Behrend* (Melbourne).

**Stampacchia, Guido. Un teorema di calcolo delle variazioni ed applicazioni a problemi al contorno per equazioni alle derivate parziali del tipo iperbolico.** Giorn. Mat. Battaglini (4) **2(78)**, 81–96 (1948).

A function $f$ defined for $(x, y)$ in a rectangle $R$ is said to satisfy condition (F) if over every rectangle $r \subset R$ the iterated integrals of $f$ exist and are equal, the integrals with respect to $x$ and to $y$ being Denjoy integrals. If $F$ exists for which $\partial^2F/\partial x\partial y=\partial^2F/\partial y\partial x=f$ for all $(x, y)$ in $R$, then $f$ satisfies condition (F). When the product of $f$ and the characteristic function of a subset $E$ of $R$ satisfies condition (F), the common value of the iterated integrals of the product defines an integral $I(f; E)$. Theorems are stated (without proof) connecting the existence of this integral over sets $G$ bounded by sectionally monotone curves $C$ with the line integrals around $C$ of $Pdx+Qdy$, where

$$P(x, y)=\int_a^y f(x, y)dy, \quad Q(x, y)=\int_a^x f(x, y)dx.$$

As application, it is stated that if $u, v$ are bounded and have all first order partials finite everywhere in $R$, and $\partial x/\partial y=\partial v/\partial x$ almost everywhere, then there exists $F$ such that $\partial F/\partial x=u$, $\partial F/\partial y=v$; also a generalization of Morera's theorem is stated. *E. J. McShane* (Charlottesville, Va.).

**Tolstov, G. P. On the mixed second derivative.** Mat. Sbornik N.S. **24(66)**, 27–51 (1949). (Russian)

The following theorems are proved. (I) There exists a function $F(x, y)$, continuous together with its first derivatives in a domain $D$, the mixed second derivatives of which exist at every point of $D$ but such that $\partial^2F/\partial x\partial y \neq \partial^2F/\partial y\partial x$ on a set $P$ of positive measure. (II) There exists a function $F(x, y)$, continuous together with its first derivatives in a domain $D$, the mixed second derivatives of which exist almost everywhere in $D$ and such that $\partial^2F/\partial x\partial y \neq \partial^2F/\partial y\partial x$ almost everywhere in $D$. (III) If the first derivatives $\partial F/\partial x$ and $\partial F/\partial y$ of the function $F(x, y)$ exist everywhere in the domain $D$, and further if $\partial F/\partial x$ is continuous in $D$ and $\partial^2F/\partial x\partial y$ exists at every point of a set of second category in $D$, then there can be found in $D$ a rectangle in which the two mixed second derivatives of $F(x, y)$ exist and are equal almost everywhere. The domain $D$ figuring in (I) is the square $0\leqq x\leqq 1$, $0\leqq y\leqq 1$ and the set $P$ is a nowhere dense perfect set of the Cantor type in $D$, the measure of which may be chosen arbitrarily near to 1. To deduce (III) the author proves the following lemma. If the function $f(x, y)$ is summable in the rectangle $a\leqq x\leqq b$, $c\leqq y\leqq d$, then for almost every $x$ in $(a, b)$ and for every $y$ in $(c, d)$ we have

$$(\partial/\partial x)\int_a^x dt\int_c^y f(t, u)du=\int_c^y f(x, u)du.$$

*A. Rényi* (Budapest).

**Tolstov, G. P. The incorrectness of Fubini's theorem for the multidimensional regular Denjoy integral.** Mat. Sbornik N.S. **24(66)**, 263–278 (1949). (Russian)

The author defines the multidimensional regular Denjoy integral by a characteristic property given by S. Kempisty [Fund. Math. **27**, 10–37 (1936), p. 33, theorem 6] and proves two propositions. (I) There exists a function $f(x, y)$ regularly integrable on $K$ $(0\leqq x\leqq 1, 0\leqq y\leqq 1)$, for which the regular integral with variable limits $F(x, y)=\iint_Q f dxdy$ (where $Q$ is the rectangle with vertices $(0, 0)$, $(x, 0)$, $(0, y)$, $(x, y)$) does not have a partial derivative $\partial F/\partial x$ except along the $x$-axis and does not have a partial derivative $\partial F/\partial y$

except along the $y$-axis. (II) There exists a function $f(x, y)$ regularly integrable in $K$ whose integral $F(x, y)$ (as above) possesses continuous first partial derivatives in $K$ but the Fubini theorem does not hold. In the example constructed to prove (II), $\partial^2F/\partial y\partial x$ exists and is equal to $f(x, y)$ in $K$, and $\partial^2F/\partial x\partial y$ exists in $K$ but differs from $\partial^2F/\partial y\partial x$ on a set of two-dimensional positive measure. [In line 23, p. 267, $\partial^2F/\partial y\partial x$ should read $\partial^2F/\partial x\partial y$.] *H. L. Smith.*

**Dubrovskiĭ, V. M. On the continuity of definite integrals which depend on a parameter.** Doklady Akad. Nauk SSSR (N.S.) **66**, 149–152 (1949). (Russian)

Let $\mathfrak{A}$ be a set of arbitrary elements and let $\mathfrak{M}$ be a class of subsets of $\mathfrak{A}$ such that $\mathfrak{M}$ contains the set $\mathfrak{A}$, the empty set, the difference of any two of its elements, and the sum of any countable sequence of its elements. The author considers integrals in the Lebesgue-Stieltjes sense, denoted by the symbol $\int_{\mathfrak{B}} f(x)M(dA_x)$, where $f(x)$ is a real-valued function on $\mathfrak{A}$, measurable relative to the class $\mathfrak{M}$ and summable relative to the nonnegative completely additive set-function $M(e)$, defined and finite on $\mathfrak{M}$, and where $\mathfrak{B}$, the domain of integration, is an element of $\mathfrak{M}$. These integrals are defined in the same way as those of Lebesgue except that the sets of $\mathfrak{M}$ take the place of measurable sets and the function $M(e)$ takes the place of measure. Set $I(\alpha)=\int_{\mathfrak{A}} f(\alpha, x)M(\alpha, dA_x)$, where, for each $\alpha$ in a set $\mathfrak{R}$, $f(\alpha, x)$, $M(\alpha, e)$ are, as functions of $x$, $e$, respectively, as above. Set $f_N(\alpha, x)=f(\alpha, x)$ if $|f(\alpha, x)|\leqq N$, $f_N(\alpha, x)=N$ if $f(\alpha, x)>N$, and $f_N(\alpha, x)=-N$ if $f(\alpha, x)<-N$. Then $I(\alpha)$ converges uniformly as to $\alpha$ if the difference $\int_{\mathfrak{A}}|f(\alpha, x)|M(\alpha, dA_x)-\int_{\mathfrak{A}}|f_N(\alpha, x)|M(\alpha, dA_x)$ converges to 0, uniformly as to $\alpha$, for $N\to\infty$. The author proves three theorems. 1. If there are positive numbers $K$ and $p>1$ such that $\int_{\mathfrak{A}}|f(\alpha, x)|^pM(\alpha, dA_x)\leqq K$, then the integral $I(\alpha)$ converges uniformly as to $\alpha$. 2. Let $\mathfrak{R}$ be metric, closed, compact, and let the functions $f(\alpha, x)$, $M(\alpha, e)$ be continuous on $\mathfrak{R}$ for each $x$ in $\mathfrak{A}$ and each $e$ in $\mathfrak{M}$. Then a sufficient condition that $I(\alpha)$ be a continuous function on $\mathfrak{R}$ is that the integral $I(\alpha)$ converge uniformly as to $\alpha$. This condition is necessary when $f(\alpha, x)$ is nonnegative. 3. If $K$, $p$ are as in theorem 1 and $f(\alpha, x)$, $M(\alpha, e)$ are continuous on $\mathfrak{R}$ for each $x$ in $\mathfrak{A}$ and each $e$ in $\mathfrak{M}$, then $I(\alpha)$ is continuous on $\mathfrak{R}$. The paper concludes with a simple example [whose statement needs to be slightly altered to make it satisfactory] to show the incorrectness of a statement by Nemyckiĭ [Rec. Math. [Mat. Sbornik] (1) **41**, 421–438 (1934), in particular, p. 436] that the continuity of $f(\alpha, x)$, $M(\alpha, e)$ in $\alpha$ and boundedness of $I(\alpha)$ imply the continuity of $I(\alpha)$. *H. L. Smith* (Baton Rouge, La.).

**Mickle, Earl J. On the extension of a transformation.** Bull. Amer. Math. Soc. **55**, 160–164 (1949).

Let $M$ be a separable metric space, let $pq$ be the distance of any two points $p, q\in M$, let $g(t)$, $0\leqq t<+\infty$, be a real-valued continuous function satisfying the following conditions: $(C')$ $g(0)=0$, $(C'')$ $g(t)>0$ for $t>0$ and $(C''')$ for any finite number of points $p_0, p_1, \cdots, p_m$ in $M$, the real quadratic form $\sum_{i,j=1}^{m}a_{ij}\xi_i\xi_j$ is positive, where

$$a_{ij}=g(p_0p_i)^2+g(p_0p_j)^2-g(p_ip_j)^2, \quad i, j=1, 2, \cdots, m.$$

A transformation $p^*=\Phi(p)$ from a set $E\subset M$ into a metric space $M^*$ will be said to satisfy the condition $C(g)$ on $E$ if, for any two points $p_1, p_2\in E$, $p_1^*p_2^*\leqq g(p_1p_2)$, where $p_i^*=\Phi(p_i)$, $i=1, 2$. The author proves that any transformation $x=\Phi(p)$, $p\in E\subset M$, from $E$ into a Euclidean space $E_n$ satisfying a condition $C(g)$ on $E$ can be extended to the whole space $M$ preserving the condition $C(g)$.

The functions $g(t)=ct$, $g(t)=ct^\alpha$, $c$ and $\alpha$ constants, with $0<\alpha\leqq 1$, satisfy the conditions $C$. Therefore the present result extends a previous one of Kirszbraun [Fund. Math. **22**, 77–108 (1934)] on Lipschitzian transformations for Euclidean spaces. *L. Cesari* (Bologna).

**Radó, Tibor. On essentially absolutely continuous plane transformations.** Bull. Amer. Math. Soc. **55**, 629–632 (1949).

Suppose $T$ maps a closed square into the plane and $A$ is the union of all essential maximal continua of constancy of $T$. Then $T$ is essentially absolutely continuous if and only if the essential multiplicity function of $T$ is summable and $T$ maps every subset of $A$ whose plane measure is zero into a set of plane measure zero. *H. Federer.*

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## *Theory of Functions of Complex Variables*

**Nöbeling, Georg. Eine allgemeine Fassung des Hauptsatzes der Funktionentheorie von Cauchy.** Math. Ann. **121**, 54–66 (1949).

L'auteur démontre le théorème fondamental de Cauchy sur les fonctions holomorphes sous la forme générale suivante. Dans le plan $E$ où l'on représente les nombres complexes $z$, on donne une courbe continue fermée $C$ de longueur finie ($C$ peut avoir une infinité de points multiples). On suppose que lorsque $z_0$ est quelconque dans $E-C$ et lorsque $z$ décrit $C$, la variation de l'argument de $z-z_0$ est bornée. Soit alors, s'il existe, $O$ l'ensemble complémentaire de $C$ dans lequel cette variation de l'argument de $z-z_0$ n'est pas nulle; si $f(z)$ est une fonction uniforme continue dans $O+C$ et holomorphe dans $O$, on a $\int_C f(z)dz=0$. L'auteur montre en outre que, au lieu de supposer $f(z)$ holomorphe dans tout $O$, il suffit de supposer $f(z)$ holomorphe dans $O-D$, $D$ étant un ensemble de points de $O$ de masse linéaire nulle au sens de Carathéodory. Dans la démonstration, l'auteur introduit comme intermédiaire l'intégrale de Lebesgue et utilise certains de ses résultats antérieurs [Jber. Deutsch. Math. Verein. **52**, 132–160 (1942), p. 159; ces Rev. **5**, 114]. *G. Valiron* (Paris).

**Tchakaloff, L. Sur les singularités polaires des séries entières.** C. R. Acad. Bulgare Sci. Math. Nat. **1**, no. 1, 9–12 (1948).

L'auteur démontre quelques théorèmes simples. Par exemple si $f(z)=\sum c_nz^n$, avec $\limsup |c_n|^{1/n}=1$, $|c_n|<M<\infty$, deux cas sont possibles: (1) il existe sur $|z|=1$ au moins une singularité non polaire, (2) $f(z)$ n'admet sur $|z|=1$ que des pôles simples. Si l'on suppose que $\limsup |c_n|=\infty$, le cas (1) est seul possible. D'ailleurs, comme l'indique l'auteur, la première hypothèse implique que tous les pôles de $f(z)$ sont simples. L'auteur indique aussi un théorème généralisant un résultat concernant les séries $\sum c_nz^{\lambda_n}$ avec $\limsup (\lambda_{n+1}-\lambda_n)\geqq k$ que nous avons démontré [S. Mandelbrojt, Ann. Sci. École Norm. Sup. (3) **40**, 413–462 (1923)]. La démonstration de son théorème doit être publiée ailleurs. [Le résultat de l'auteur est contenu dans des recherches récentes de S. Agmon, C. R. Acad. Sci. Paris **226**, 1786–1787 (1948); ces Rev. **9**, 576.] *S. Mandelbrojt.*

**Ilieff, Ljubomir. Analytisch nichtfortsetzbare Potenzreihen.** C. R. Acad. Bulgare Sci. Math. Nat. 1, no. 1, 25–28 (1948).

Soit $f(z)=\sum\gamma_n c_n z^n$ avec (1) $\lim c_{n+1}/c_n=1$,

(2) $\quad 0<\liminf |c_n|/n^n\leqq\limsup |c_n|/n^n<\infty$,

(3) les termes de la suite $\{\gamma_n\}$ ne prennent qu'un nombre fini de valeurs $\delta_1, \cdots, \delta_s$, $s>1$; si les $\gamma_n$ ne se reproduisent pas, pour $n$ assez grand, périodiquement, le cercle $|z|=1$ est une coupure pour $f(z)$. Ce théorème [publié par l'auteur dans l'Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 41, 31–42 (1944)] est ensuite généralisé de la manière suivante. La suite $\{\gamma_n\}$ est partagée en $s$ suites partielles admettant chacune une limite. Supposons que la $r$ième suite ($r=1, 2, \cdots, s$) ($s>1$) admette $\alpha_r$ comme limite. Supposons que les $\alpha_r$ soient distincts, la suite $\{\gamma_n\}$ n'admettant aucun autre point limite que les $\alpha_r$. Désignons par $K_n$ le nombre $r$ tel que $\gamma_r$ appartienne à la $r$ième suite. Le théorème précédent peut être généralisé de la manière suivante. Si (1), (2) sont satisfaits et si, pour $n$ assez grand, la suite n'est pas périodique le cercle $|z|=1$ est une coupure. Deux autres théorèmes fournissant des séries à coupure sont indiqués. Le dernier théorème est lié au théorème que nous avons démontré en 1923 [voir l'analyse ci-dessus (Tchakaloff)]. Remarquons, toutefois, que les résultats d'Ilieff résultent de ceux publiés récemment par S. Agmon [C. R. Acad. Sci. Paris 226, 1497–1499, 1673–1674, 1786–1787, 1875–1876 (1948); ces Rev. 9, 576].

*S. Mandelbrojt* (Houston, Tex.).

**Métral, Paul. Fonctions analytiques presque loxodromiques et presque périodiques loxodromiques. Généralisations-relation avec les problèmes de l'itération.** C. R. Acad. Sci. Paris 228, 1557–1559 (1949).

Inside a strip the author considers analytic functions of a certain type called by him "almost periodic loxodromic" [cf. same C. R. 227, 752–753 (1948); these Rev. 10, 250]. It seems to the reviewer that the only such functions are constants.

*E. Følner* (Copenhagen).

**Beurling, Arne. Some theorems on boundedness of analytic functions.** Duke Math. J. 16, 355–359 (1949).

Let $M(r)=\max_\theta |f(re^{i\theta})|$ and $m(r)=\min_\theta |f(re^{i\theta})|$ for an entire function $f$. A conjecture of Wiman [Acta Math. 41, 1–28 (1916)] states that for $k<1$ the inequality $M(r)>[m(r)]^{-k}$ has arbitrarily large solutions. In this paper the conjecture of Wiman is established for the case where the minimum modulus is attained on a ray. The proof is a consequence of the following theorem. Let $f$ be analytic for $|z|\leqq R$ and let $\epsilon$ and $C$ denote positive constants. Let $\alpha(\epsilon)$ denote the least positive root of $[\cos 2\alpha]^{\pi/2\alpha}=(1+\epsilon)^{-1}$. If $|f(r)|[M(r)]^{1+\epsilon}\leqq C^{2+\epsilon}$ for $0\leqq r\leqq R$, then $|f(z)|\leqq C$ in the region $D_{R,\alpha}$ containing the interval $(0, R)$ and bounded by the two logarithmic spirals with pole at $z=0$ which pass through $z=R$ and make an angle $\alpha(\epsilon)$ with their radius vectors. In particular, $D_{R,\alpha}$ contains the circle $|z|<Re^{-\pi^2(1+\epsilon)/\epsilon}$. The proof is based on the use of a logarithmic transformation, the Poisson integral for a half-plane, and the principle of the harmonic majorant. Two related theorems are also given. (1) Let $f(z)$ be entire and $\epsilon(r)$ nonincreasing for $r\geqq 0$ and such that

$$\gamma=\limsup_{r\to\infty} \pi^{-1}\epsilon(r)\log r>0.$$

If $|f(r)|[M(r)]^{1+\epsilon(r)}$ is bounded for $r\geqq 0$, then $f$ is bounded in the sector $|\arg z|<\gamma$. If $\gamma\geqq\pi$, $f$ is constant. (2) Let $\epsilon(r)$ and $\gamma$ be as in (1). Let $f$ be analytic for $0<\arg z<2\beta$ ($\leqq 2\pi$) and regular at all finite boundary points. Let $M(r)=\max |f(re^{i\theta})|$ for $0\leqq\theta\leqq 2\beta$. If

$$|f(r)|[M(r)]^{1+\epsilon(r)}\leqq C^{2+\epsilon(r)}$$

for $r\geqq 0$, then $|f(z)|\leqq C$ for $0<\arg z<\min(\beta, \gamma)$.

*M. Heins* (Providence, R. I.).

**Littlewood, J. E., and Offord, A. C. On the distribution of zeros and $a$-values of a random integral function. II.** Ann. of Math. (2) 49, 885–952 (1948); errata, 50, 990–991 (1949).

[Bezüglich Teil I, s. J. London Math. Soc. 20, 130–136 (1945); diese Rev. 8, 372.] G. Pólya [Math. Z. 29, 549–640 (1929)] hatte gezeigt, dass "fast alle" ganzen Funktionen unendlich hoher Ordnung sämtliche Richtungen zu Julia'schen Richtungen haben, und er hatte die Vermutung ausgesprochen, dass dies auch für "fast alle" ganzen Funktionen positiver endlicher Ordnung zutreffe, was dann von A. Dvoretzky bewiesen wurde. Die Verf. haben sich in der vorliegenden Arbeit die Aufgabe gestellt, die genannten Ergebnisse im Falle positiver endlicher Ordnung wesentlich zu verschärfen und zu ergänzen.

Zur Präzisierung von "fast alle" wird eine ganze Funktion $F(z)=\sum_0^\infty a_n z^n$ gegeben und dieser eine Klasse $\mathfrak{F}$ von ganzen Funktionen zugeordnet, indem die Faktoren $+1$ und $-1$ zufällig auf die Koeffizienten an verteilt werden. Mit Hilfe der Rademacher'schen Funktionen $r_n(t)$ wird dann jedem von $p2^{-q}$ verschiedenen $t$-Wert im Intervall $(0, 1)$ umkehrbar eindeutig eine Funktion $f(z, t)=\sum_0^\infty r_n(t)a_n z^n$ der Klasse $\mathfrak{F}$ zugeordnet. Damit kann jeder Teilmenge von $\mathfrak{F}$ ein Mass zugeschrieben werden, das gleich dem Mass der entsprechenden $t$-Menge ist.

Von den Verf. werden unter anderm folgende Resultate bewiesen. Bei fast allen $f\in\mathfrak{F}$ (Mass der $t$-Menge $=1$) sind sämtliche Richtungen sogar Borel'sche Richtungen. Genauer, sind $r_1(a), r_2(a), \cdots$ die Beträge der $a$-Stellen von $f$ in $|\arg z-\alpha|<\epsilon$, $\psi(x)>0$ eine abnehmende Funktion mit $\psi(2x)=O(\psi(x))$ und $\sum_1^\infty \psi(n)n^{-1+\delta}<\infty$ für ein $\delta>0$, so sind bei fast allen $f\in\mathfrak{F}$ die Reihen $\sum^\infty\psi(r_n(a))$ und das Integral $\int^\infty\psi(u)dN(u)$ für alle $a$, $\alpha$, und $\epsilon>0$ gleichzeitig konvergent oder divergent. Dabei bezeichnet $N(R)$ den Maximalindex in der Reihe $\sum_1^\infty |a_n|R^n$. Die Pólya'sche Vermutung kann auch in dieser Weise verschärft werden, dass man die Winkelräume $|\arg z-\alpha|<\epsilon$ durch asymptotisch wesentlich engere Gebiete ersetzt. So nehmen z. B. fast alle $f\in\mathfrak{F}$ in jedem Parallelstreifen jeden Wert unendlich oft an, sobald die Ordnung $\rho>1$ ist.

In der von den Verf. entwickelten Theorie spielen der Maximalterm $m(R)$ und der Maximalindex $N(R)$ in der Reihe $\sum_0^\infty |a_n|R^n$ und andere mit $m(R)$ und $N(R)$ gebildete Grössen sowie der Begriff des "pit" eine wesentliche Rolle. Die "pits" sind kleine Umgebungen der Nullstellen von $f$, auf deren Rand $\log|f(z)|$ von der Grössenordnung $\log m(|z|)-C|z|^{\frac{1}{2}}$ ($C$ eine numerische Konstante) ist. Ein "pit" wird so oft gezählt, als es Nullstellen enthält. Ein Hauptresultat der Arbeit bezieht sich auf die Verteilung dieser "pits." Zu jedem $\epsilon>0$ gibt es ein $R_0$ und eine Teilmenge $\mathfrak{F}^*\subset\mathfrak{F}$ vom Mass $>1-\epsilon$, die beide nur von $\epsilon$ und den $|a_n|$ abhängen, sodass für jedes $|z|=R>R_0$ und $f\in\mathfrak{F}^*$ gilt: (a) Ein "pit" im Abstand $R$ von 0 hat einen Durchmesser $\leqq 2e^{-\frac{1}{2}R^2}$ und $f$ nimmt darin jeden Wert $a$ mit $\log|a|<\log m(R)-8(3R)^{\frac{1}{2}}$ gleich oft an; (b) ausserhalb der "pits" ist $\log^+|f(z)|>\log m(|z|)-8(3|z|)^{\frac{1}{2}}$; (c) die Anzahl der "pits" in einem Sektor $|\arg\zeta-\alpha|\leqq\gamma$, $|\zeta|\leqq\rho$

liegt zwischen $(\gamma/1000\pi)N((1-\gamma)\rho)-\rho^\varepsilon/\gamma$ und

$$2\gamma N((1+\gamma)\rho)+\rho^\varepsilon/\gamma,$$

ihre Anzahl im Kreisring $\rho-r\leqq|z|\leqq\rho+r$ zwischen $(1/40\pi)N_*(\rho,r)-\rho^{1+\varepsilon}/r$ und $6N^*(\rho,r)+\rho^{1+\varepsilon}/r$, und ihre Anzahl im Kreis $|\zeta-z|<r$ ($|z|=R$) zwischen

$$(1/40\pi)(r/R)N_*(R,r)-R^\varepsilon$$

und $2(r/R)N^*(R,r)+R^\varepsilon$. Dabei ist

$$N_*(R,r)=\max_{0\leqq\varkappa\leqq1}(1-\varkappa)^{\frac{1}{2}}(N(R+\varkappa r)-N(R-\varkappa r))$$

und $N^*(R,r)$ eine analog gebildete Grösse. Die angegebenen Schätzungen hängen also im Wesentlichen nur von den $|a_n|$ ab. Die angegebenen Resultate sind in gewissem Sinne die bestmöglichen. [In Formel (1.4.2) streiche den zweiten Exponenten $\frac{1}{2}$.] *A. Pfluger* (Zürich).

**Lax, Peter D. The quotient of exponential polynomials.** Duke Math. J. **15**, 967–970 (1948).

Ist der Quotient zweier Exponentialpolynome $\sum_1^n A_\nu e^{\alpha_\nu z}$ eine ganze Funktion, so ist dieser Quotient wieder ein Exponentialpolynom. Von diesem Spezialfall eines Theorems von J. F. Ritt [Trans. Amer. Math. Soc. **31**, 680–686 (1929)] gibt Verf. einen neuen Beweis mit Hilfe des Phragmén-Lindelöf'schen Prinzips. *A. Pfluger* (Zürich).

**Shah, S. M. A note on uniqueness sets for entire functions.** Proc. Indian Acad. Sci., Sect. A. **28**, 519–526 (1948).

Es sei $0<\rho<\infty$, $n(r)$ die Anzahlfunktion der Stellen $a_n$, $n=1, 2, \cdots$, $|a_n|\to\infty$ und

$$\limsup r^{-\rho}n(r)=L,\quad \liminf r^{-\rho}n(r)=l>0.$$

Es bezeichne $C(T,t)$ die Klasse der ganzen Funktionen $f(z)$, für welche $f(a_n)=0$, $n=1, 2, \cdots$ und $\limsup r^{-\rho}\log M(r)=T$ und $\liminf r^{-\rho}\log M(r)=t$ ist. Verf. beweist, dass im Falle $t<l/\rho$ oder $T<(e\rho)^{-1}Le^{l/L}$, die Klasse $C(T,t)$ aus dem einzigen Element 0 besteht. Ein entsprechend schärferes Resultat ergibt sich im Falle, dass alle $a_n$ in einem Winkelraum liegen. Dies sind Verfeinerungen analoger Sätze von Boas [Ann. of Math. (2) **47**, 21–32 (1946); **48**, 1085 (1947); diese Rev. **7**, 425; **9**, 180], wo nur der Typus $T$ der ganzen Funktion $f$ in Betracht gezogen wird. *A. Pfluger* (Zürich).

**Buck, R. Creighton. Integral valued entire functions.** Duke Math. J. **15**, 879–891 (1948).

Verf. stellt sich die Aufgabe, jene ganzwertigen Funktionen zu bestimmen, die in gewissen Klassen ganzer Funktionen vom Exponentialtypus enthalten sind. Mit Hilfe einer allgemeinen Methode, welche von G. Pólya [vgl. sein Referat zur Arbeit von A. Selberg, Arch. Math. Naturvid. **44**, 45–52 (1941); diese Rev. **2**, 356] angeregt und von C. Pisot [Jber. Deutsch. Math. Verein. **52**, 95–102 (1942); diese Rev. **4**, 270; **6**, 334] verwertet wurde, werden mannigfache Spezialfälle diskutiert, welche die frühern in diesem Problemkreis erzielten Resultate von G. Pólya, F. Carlson, A. Selberg und C. Pisot enthalten. *A. Pfluger* (Zürich).

**Buck, R. Creighton. Interpolation series.** Trans. Amer. Math. Soc. **64**, 283–298 (1948).

Verf. behandelt Konvergenz-, Summabilitäts-, und Eindeutigkeitsfragen bei Interpolationsreihen für ganze Funktionen mit Hilfe einer allgemeinen [Proc. Nat. Acad. Sci. U. S. A. **33**, 288–292 (1947); diese Rev. **9**, 232] veröffentlichten Methode. Es sei $f(z)$ eine ganze Funktion vom Exponentialtypus, $\phi(w)$ ihre Borel'sche Transformierte, $D(f)$ die konvexe Hülle der Singularitäten von $\phi(w)$, und $\{g_n(w)\}$ eine Folge von ganzen Funktionen. Dieser Folge entspricht eine Folge linearer Funktionale

$$T_n(f)=(2\pi i)^{-1}\oint g_n(w)\phi(w)dw.$$

Ist $u_n(z)$ eine zu $\{T_n\}$ orthogonale Folge, d.h. $T_m(u_n)=\delta_{m,n}$, so kann jedem $f$ die sogenannte Interpolationsreihe für $\{T_n\}$ zugeordnet werden, $f(z)\sim\sum_0^\infty T_n(f)\cdot u_n(z)$. Bei gegebener Folge $\{g_n(w)\}$ diskutiert Verf. die Klassen jener $f$, für welche die Reihe konvergiert oder $E$-summierbar ist zu $f$:

$$(E)\text{-}\lim s_n=\lim_{\lambda\to\infty}\{\sum_0^\infty d_n s_n\lambda^n/\sum_0^\infty d_n\lambda^n\},$$

bzw. welche durch die Folge $T_n(f)$ eindeutig bestimmt sind. Sind $g_n=(\zeta(w))^n$, $\Omega_w$ das Univalenzgebiet von $\zeta(w)$, das den Nullpunkt enthält, $\Omega_\zeta$ sein Bild vermittels $\zeta=\zeta(w)$, $\Delta_\zeta$ der grösste darin enthaltene Kreis $|\zeta|<R$ und $\Omega_\zeta^*$ der $E$-Stern von $\Delta_\zeta$, sind ferner $\Delta_w$ und $\Omega_w^*$ ihre Urbilder in $\Omega_w$, so gilt das folgende allgemeine Theorem: Die Interpolationsreihe konvergiert zu $f$, wenn $D(f)\subset\Delta_w$ und ist $E$-summierbar zu $f$, wenn $D(f)\subset\Omega_w^*$. Dieses Theorem wird auf verschiedene spezielle Reihen angewandt, die Newton'sche und Stirling'sche Reihe, auf allgemeine Differenzenreihen, Abel'sche und Lidstone'sche Reihen. Es soll nur im Falle der Newton'schen Reihe das diesbezügliche Resultat erwähnt werden: für $\zeta(w)=e^w-1$ wird $T_n(f)=\Delta^n f(0)$, $u_n(z)=\{z(z-1)\cdots(z-n+1)\}/n!=C_{z,n}$, $\Omega_w$ der Streifen $|v|<\pi$, $\Delta_w$ das Gebiet $|e^w-1|<1$ in $\Omega_w$. Die Reihe $\sum_0^\infty\Delta^n f(0)\cdot C_{z,n}$ konvergiert dann, sobald $D(f)\subset\Delta_w$ und ist Mittag-Leffler-summierbar, sobald $D(f)\subset\Omega_w$. Alle $f$, für welche Mittag-Leffler-Summabilität vorliegt, sind durch die $T_n(f)$ eindeutig bestimmt. *A. Pfluger* (Zürich).

**Lohin, I. F. Concerning the representation of entire analytic functions.** Doklady Akad. Nauk SSSR (N.S.) **66**, 157–160 (1949). (Russian)

The author's results are all included in those of R. C. Buck [see the preceding review and the reference given there]. *R. P. Boas, Jr.* (Providence, R. I.).

**Levin, B. On functions of finite degree, bounded on a sequence of points.** Doklady Akad. Nauk SSSR (N.S.) **65**, 265–268 (1949). (Russian)

It was proved by Duffin and Schaeffer [Amer. J. Math. **67**, 141–154 (1945); these Rev. **6**, 148] that an entire function $G(z)$ of exponential type [Russian, "finite degree"] less than $\pi$ is bounded on the real axis if it is bounded on a real sequence $\{\lambda_k\}$ with $|k-\lambda_k|<L<\infty$ and $|\lambda_k-\lambda_n|\geqq\delta>0$ for $k\neq n$. The author, evidently without knowing of the work of Duffin and Schaeffer, proves a somewhat more general result (assuming only that $h(\pi/2)+h(-\pi/2)<2\pi$, where $h(\theta)$ is the indicator function of $G(z)$, and allowing $\lambda_k$ to be complex) by a quite different method, based on interpolation series. ~~He also can obtain the corresponding theorem for a half-plane.~~ *R. P. Boas, Jr.*

**Ahiezer, N. I. On interpolation of entire transcendental functions of finite degree.** Doklady Akad. Nauk SSSR (N.S.) **65**, 781–784 (1949). (Russian)

The author generalizes the interpolation formula used by Levin in the paper reviewed above. Let $G(z)$ and its zeros $\lambda_k$ satisfy the following conditions: $G(z)$ is of exponential type $\pi$; $|G(x+iy_0)|\leqq A|x-i|^\alpha$ $(-\infty<x<\infty)$; $|G(\pm iy)|\geqq By^\beta e^{\pi y}$ $(y\geqq y_1>0)$; $|G'(\lambda_k)|\geqq C\{1+|\lambda_k|\}^\gamma$

($k=0, \pm1, \pm2, \cdots$); $|\Im\lambda_k|<y_2$, where the unspecified letters are constants. Let $f(z)$ be an entire function of exponential type for which, for some nonnegative integer $q$ and $\rho<\frac{1}{2}$,

$$\sum_{k=-\infty}^{\infty}\frac{|f(\lambda_k)|^2}{|G'(\lambda_k)|^2|\lambda_k|^{2q+2\rho}}<\infty,$$

$$\int_{-\infty}^{\infty}e^{-2x|y|}|f(iy)|^2\frac{dy}{1+|y|^{2q+2\beta}}<\infty,$$

$$\int_1^{\infty}u^{-2}\log\int_u^{\infty}e^{-xy}|f(\pm iy)|\,y^{-q-\beta-2}dydu=-\infty.$$

Then

$$f(z)=\sum_{k=-\infty}^{\infty}{}'\frac{G(z)f(\lambda_k)}{G'(\lambda_k)(z-\lambda_k)}\left(\frac{z}{\lambda_k}\right)^q+z^{-1}G(z)P(z),$$

where $P(z)$ denotes the sum of the first $q+1$ terms of the Maclaurin series of $zf(z)/G(z)$. In Levin's work

$$G(z)=z\prod_{n=1}^{\infty}(1-z/\lambda_n)(1-z/\lambda_{-n}).$$

The author mentions the following application in addition to the theorem of the preceding review. Let

$$\lim y^{-1}f(\pm iy)e^{-xy}=0,\quad z^{-1}f(z)e^{-x|z|}=O(1),$$

$|f(k)|\leqq M$ for all integers $k$. If there are numbers $N$ and $n$ such that in every interval of length $n$ there is an integer $m$ with $|f'(m)|<N$, then $f(z)$ is bounded on the real axis.

*R. P. Boas, Jr.* (Providence, R. I.).

*Korevaar, Jacob. **Approximation and Interpolation Applied to Entire Functions.** Thesis, University of Leiden, 1949. viii+56 pp.

The work consists of two parts. A sequence of polynomials is a $C_1$-sequence if it converges everywhere, and uniformly in every bounded region. Laguerre was perhaps the first to show a connection between the distribution of zeros of the polynomials of a $C_1$-sequence and the properties of the limit (entire) function. Further work in this direction is due to Pólya, Obrechkoff, Szász, Weisner and others. [One may cite the monograph of N. Obrechkoff, Quelques classes de fonctions entières limites de polynomes et de fonctions méromorphes limites de fractions rationnelles, Actual. Sci. Ind., no. 891, Hermann, Paris, 1941; these Rev. 7, 516, for an exposition of some of these results.] Part I states the principal results referred to above, and proves a number of generalizations (which are too complicated in statement to cite here). These lead the author to pose and solve the following general problem. Given a point set $R$ in the complex plane; to characterize the corresponding class of $R$-functions; i.e., those entire functions each of which is the limit of a $C_1$-sequence of polynomials $\{f_n(z)\}$, with the zeros of $f_n(z)$ in $R$ for all $n$.

If $R$ is bounded, then $f(z)\not\equiv0$ is an $R$-function if and only if $f(z)$ is a polynomial with all zeros in $\bar{R}$ (the closure of $R$). Suppose then that $R$ is unbounded. The angle $\varphi$ is called an asymptotic direction of $R$ if there exists a sequence $\{\zeta_n\}$ in $R$ with $|\zeta_n|\to\infty$, $\arg\zeta_n\to\varphi$ ($n\to\infty$); and the set of points $\exp(i\varphi)$, where $\varphi$ runs through all asymptotic directions, is denoted by $D$ ($=D_R$). Three cases are distinguished. (I) A positive integer $j$ exists such that $D^j$ (the set of $j$th powers of the points of $D$) belongs to an arc of length less than $\pi$. Let $j=k$ be the smallest value for which this is true. It is supposed that for no $j$ in $1\leqq j<k$ does $D$ belong to an arc of length $\pi$. (II) A positive integer $j$ exists such that $D^j$ belongs to an arc of length $\pi$. Let $j=v$ be the smallest such $j$. It is supposed that $D^v$ belongs to no arc of length less than $\pi$. (III) For no positive integer $j$ does $D^j$ belong to an arc of length $\pi$.

In case (I) $f(z)$ is an $R$-function if and only if

(*) $$f(z)=Ae^{a_1z+\cdots+a_{k-1}z^{k-1}}z^m\prod_p\left(1-\frac{z}{z_p}\right)\exp\left(\frac{z}{z_p}+\cdots+\frac{z^{k-1}}{(k-1)z_p^{k-1}}\right),$$

where $A, a_1, \cdots, a_{k-1}$ are arbitrary complex numbers; $a_k$ satisfies the condition that $\exp\{-i\arg(-a_k)\}$ belongs to the closed arc of length less than $\pi$ bounded by the extremal points of $D^k$; $m\geqq0$ if 0 is a point of $\bar{R}$, and $m=0$ otherwise; and $z_p\in\bar{R}$, $\sum_p|z_p|^{-k}<\infty$. In case (III), the class of $R$-functions (excluding $f(z)\equiv0$) is identical with the class of entire functions all zeros of which lie in $\bar{R}$. Only a partial solution is obtained in case (II). It is shown that if, in addition to $D^j$'s lying on an arc of length $\pi$ (for some $j$), it is also true that $R^t$ belongs for some $t$ to a half-plane whose boundary passes through 0, then every $R$-function, not identically zero, has the form (*) with $k=2t$, where $m\geqq0$ and $\sum_p|z_p|^{-2t}<\infty$. (A further convergence condition, too lengthy to quote, is also fulfilled.) If, further, $R^s$ belongs for some $s$ to a sector of opening less than $\pi$, and if $f(z)(\not\equiv0)\in R$, then $f(z)$ has the form (*) with $k=\min(2t, s)$, where $\sum_p|z_p|^{-k}<\infty$ and $m\geqq0$. On the other hand, an example is given of case (II) in which every entire function whose zeros belong to $\bar{R}$ is an $R$-function. Part I ends with the consideration of a number of particular sets $R$.

Part II poses a different problem. S. Bernstein showed [C. R. Acad. Sci. Paris 176, 1603–1605 (1923)] that if an entire function $f(z)$ of exponential type zero is bounded on a line, then $f(z)\equiv$constant. Pólya replaced the line condition by the point set $\{n\}$, $n=0, \pm1, \pm2, \cdots$ [Aufgabe 105, Jber. Deutsch. Math. Verein. 40, italic p. 80 (1931)], and since then numerous improvements and generalizations have been given. [For a report on this subject, see R. P. Boas, Bull. Amer. Math. Soc. 48, 839–849 (1942); these Rev. 4, 136.] These results center around the following formulation. If $g(z)$ is an entire function of exponential type, to find a class of entire functions $f(z)$ of exponential type, sufficiently small relative to $g(z)$ such that if $f(z)$ is bounded on the set of zeros of $g(z)$, then $f(z)$ is a constant. The author states many of the known results and improves a number of them. Of his numerous theorems we cite three. (i) Let $g(z)$ be entire and of exponential type, with zeros $\{z_n\}$. Let $U$ be a double sector $|z|\geqq0$, $\varphi_1\leqq\arg z\leqq\varphi_2$, $\varphi_1+\pi\leqq\arg z\leqq\varphi_2+\pi$ ($\varphi_1<\varphi_2$). To every $d>0$ let positive constants $A, B$ exist such that $|g(z)|>Be^{A|z|}$ for all $z\in U$ for which $|z-z_n|\geqq d$; and let $C>0$ exist such that $|g'(z_n)|>C$ ($n=1, 2, \cdots$). If $f(z)$, entire and of exponential type, is bounded on the set $\{z_n\}$, and if $f(z)$ is of zero type on some line lying in $U$, then $f(z)\equiv$constant. (ii) Let $f(z)$ be regular and of exponential type in the sector $S$: $|z|\geqq0$, $\varphi_1\leqq\arg z\leqq\varphi_2$ ($\varphi_1<0<\varphi_2$, $\varphi_2-\varphi_1\leqq\pi$). For some positive integer $k$ and positive $C$, let $|f^{(n)}(n)|<C$ ($s=0, 1, \cdots, k-1$; $n=1, 2, \cdots$). If $f(z)$ is of smaller type than $\sin^k\pi z$ on the boundary of $S$ [i.e., positive $\beta$, $\delta$ exist such that

$$|f(re^{i\varphi_j})|<\beta\exp\{(k\pi-\delta)|\sin\varphi_j|r\}$$

($r\geqq0$, $j=1, 2$)], then $f(z)$ is bounded on the positive real axis. (iii) Let $g(z)$ be entire and of exponential type, whose zeros $\{z_n\}$ satisfy $|z_n-z_m|\geqq c$ ($n\neq m$, for some $c>0$). Let $\alpha>0$, $\beta>0$, $\varphi_0$ exist such that to every $\epsilon>0$, $d>0$ there

correspond $A(\epsilon)>0$, $B(\epsilon,d)>0$, $B'(d)>0$ for which

$$|g(re^{i\varphi})|<A(\epsilon)\exp\{(\alpha+\epsilon)r\},$$
$$|g(re^{i\varphi})|>B(\epsilon,d)\exp\{(\alpha-\epsilon)r|\cos(\varphi-\varphi_0)|\},$$
$$|g(re^{i\varphi})|>B'(d)\exp(\beta r)$$

(the last two inequalities under the restriction $|re^{i\varphi}-z_n|\geqq d$). Let $f(z)$, entire and of exponential type, be bounded on set $\{z_n\}$. Let $U$ be a double sector (as in (i) above). If to every $d>0$ there exist positive $D(d)$, $E(d)$ such that

$$|f(z)|<D(d)e^{-E(d)|z|}\cdot|g(z)|$$

for all $z\in U$ for which $|z-z_n|\geqq d$, then $f(z)\equiv$constant.

*I. M. Sheffer* (State College, Pa.).

**Korevaar, J. The zeros of polynomials converging to an entire function and its canonical representation.** Math. Centrum Amsterdam. Rapport **ZW 1948-017**, 18 pp. (1948). (Dutch. English summary)

The results of this report are treated in more detail in the first part of the author's thesis reviewed above.

*R. P. Boas, Jr.* (Providence, R. I.).

**Ibragimov, I. I. On the interpolation of entire periodic functions.** Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 231–244 (1949). (Russian)

The following theorem is proved. Let $a_n=\alpha_n+i\beta_n$, $-\pi\leqq\alpha_n<\pi$, be a given set of points, and $n(\log r)$ the number in the upper (lower) half plane inside the rectangle with sides $x=\pm\pi$, $y=\pm\log r$. Let $M^{\pm}(\log r)$ be the maximum modulus on $y=\pm\log r$ of the periodic entire function $f(z)$ of period $2\pi$. Then the interpolation series

$$f(z)=\sum_{m=-\infty}^{\infty}\frac{\prod_{k=-n}^{n}\sin\frac{1}{2}(z-a_k)}{\prod_{k=-n,k\neq m}^{n}\sin\frac{1}{2}(a_m-a_k)}\frac{f(a_m)}{\sin\frac{1}{2}(z-a_m)}$$

converges uniformly in every finite region if both inequalities $\log M^{\pm}(\log(r/\theta))\leqq c(\theta)n(\log r)$ are satisfied with $c(\theta)<\log\{(1-\theta)/\theta\}$, $0<\theta<\frac{1}{2}$. Two corollaries are given. Define $\rho^+$ and $\rho^-$ by $\rho^{\pm}=\limsup_{d\to\infty}d^{-1}\log\log M^{\pm}(d)$, and if they are finite, define $\sigma^+$, $\sigma^-$ by

$$\sigma^{\pm}=\limsup_{d\to\infty}\exp(-d\rho^{\pm})\log M^{\pm}(d).$$

(1) If $\rho^{\pm}<1$, or if $\rho^{\pm}=1$ and $\sigma^{\pm}<0.278\cdots$ (the positive root of $\lambda e^{\lambda+1}=1$), then $f(z)$ is represented by the series above with $a_m=i\log m$ $(m=2,3,\cdots)$, $a_{-m}=-i\log m$ $(m=1,2,\cdots)$. (2) If the interpolation points satisfy

$$\limsup_{n\to\infty}\log(\beta_n/n^{1/\rho^{\pm}})=\omega^{\pm},$$

the function $f(z)$ is represented by the interpolation series provided $\sigma^{\pm}<\exp(-\rho^{\pm}\omega^{\pm})\lambda^{\rho^{\pm}}(\lambda+1)^{1-\rho^{\pm}}/\rho^{\pm}$, with $\lambda$ the positive root of $\lambda^{\rho^{\pm}}e^{\lambda+1}=1$. The limitation $\theta<\frac{1}{2}$ in the main theorem is best possible in the following sense. If $\frac{1}{2}<\theta<1$, and $\epsilon>0$, then there are an entire $F(z)$ of period $2\pi$, such that $\log M^{\pm}(d)<\exp(d^{1+\epsilon})+c$, and a sequence of points $a_n$ with $\lim n(\log\theta d)/\log M^{\pm}(\log d)=\infty$, such that the interpolation series diverges.

*R. P. Boas, Jr.*

**Leont'ev, A. F. On interpolation in the class of entire functions of finite order and normal type.** Doklady Akad. Nauk SSSR (N.S.) **66**, 153–156 (1949). (Russian)

The author extends his earlier results [Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 33–44 (1949); same Doklady (N.S.) **61**, 785–787 (1948); these Rev. **10**, 602, 289]. Let $m$ be the smallest integer exceeding $\rho$, and $\epsilon=e^{2\pi i/m}$. Let $\{\lambda_n\}$ be a sequence of complex numbers arranged according to increasing modulus. In order that, for every system $\{a_{n,p}\}$, $p=0,1,\cdots,m-1$, such that $\limsup\log|a_{n,p}|\cdot|\lambda_n|^{-\rho}<\infty$, there should exist an entire function $\omega(z)$ of growth not exceeding order $\rho$, finite type, with $\omega(\epsilon^p\lambda_n)=a_{n,p}$, it is necessary and sufficient that $\{\lambda_n\}$ satisfy the conditions

(A) $$\limsup n/|\lambda_n|^{\rho}=\sigma<\infty,$$

(B) $$\limsup|\lambda_n|^{-\rho}\log|1/F'(\lambda_n)|=\delta<\infty,$$

$$F(z)=\prod_{n=1}^{\infty}(1-z^m/\lambda_n^m).$$

Some supplementary results are deduced as by-products.

*R. P. Boas, Jr.* (Providence, R. I.).

**Leont'ev, A. F. On an interpolation problem.** Doklady Akad. Nauk SSSR (N.S.) **66**, 331–334 (1949). (Russian)

Let $\{\lambda_n\}$ be an increasing sequence of positive real numbers, such that $\limsup n/\lambda_n^{\rho}<\infty$, $\rho>0$. If there is a function $f(z)$ regular in some angle $|\arg z|<\mu$, $\mu>0$, and satisfying $|f(z)|<\exp(c|z|^{\rho})$ for large $|z|$ in the angle, with the property $f(\lambda_n)=(-1)^n$, then for every set $\{a_n\}$ such that $\limsup\lambda_n^{-\rho}\log|a_n|<\infty$ there is an entire function $\omega(z)$ of growth not exceeding order $\rho$, finite type, such that $\omega(\lambda_n)=a_n$. Hence the author deduces a necessary and sufficient condition on $\{\lambda_n\}$ for such an entire function to exist.

*R. P. Boas, Jr.* (Providence, R. I.).

**Leont'ev, A. F. On functions represented by series of Dirichlet polynomials.** Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 221–230 (1949). (Russian)

Let $\{\lambda_n\}$ be a sequence of complex numbers, with $\sigma=\limsup n/|\lambda_n|<\infty$. The author considers properties of a function $f(z)$ which is represented by a uniformly convergent series of functions $P_n(z)=\sum_{j=1}^{n}\alpha_{n,j}e^{\lambda_j z}$ in a set $D$ containing a circle of radius $r>\pi\sigma$. [He has previously considered the problem in more detail for $\lambda_n\geqq 0$ in the case when $\lim n/\lambda_n$ exists; his results were outlined in Uspehi Matem. Nauk (N.S.) **3**, no. 4(26), 176–180 (1948); these Rev. **10**, 364, but proofs have not been made available in any publication circulated outside the USSR.] (1) The region consisting of those regular points of $f(z)$ which can be reached from the circle mentioned above without going within $\pi\sigma$ of a singular point is simply connected and one-sheeted. (2) If $\sigma=0$, the domain of regularity of $f(z)$ is convex. (3) If the sequences $P_n(z)$ and $P_n'(z)=\sum_{j=1}^{n}\alpha'_{n,j}e^{\lambda_j z}$ both converge uniformly to $f(z)$ in $D$, then

$$\lim_{n\to\infty}\alpha_{n,j}=\lim_{n\to\infty}\alpha'_{n,j}=\alpha_j.$$

Hence the Dirichlet series $\sum\alpha_j e^{\lambda_j z}$ converges to $f(z)$ if it converges uniformly in $D$. A final theorem assures the convergence of this Dirichlet series if $f(z)$ is regular in a certain region. [Cf. Mandelbrojt, Trans. Amer. Math. Soc. **55**, 96–131 (1944); these Rev. **5**, 176.]

Put

$$L_{k,n}(z)=\prod_{j=k}^{n}(1-z^2/\lambda_j^2)=\sum(a_{n,k,j}/j!)z^j,$$

$$L_{k,\infty}(z)=\prod_{j=k}^{\infty}(1-z^2/\lambda_j^2)=\sum(a_{k,j}/j!)z^j$$

and define operators $M_{k,n}$ and $M_{k,\infty}$ by replacing $z^j$ by $f^{(j)}(z)$ in $L_{k,n}$, $L_{k,\infty}$. Then

(a) $$M_{k,\infty}(e^{\lambda z})=e^{\lambda z}L_{k,\infty}(\lambda);$$

(b) $$M_{k,\infty}(f)=M_{k,n}\{M_{n+1,\infty}(f)\};$$

(c) if $f_n(z)\to f(z)$ uniformly in $|z-z_0|<\rho$, where $\rho>\pi\sigma$, then $M_{k,\omega}(f_n)\to M_{k,\omega}(f)$ in a neighborhood of $z_0$; (d) in the component of $D$ which contains $z_0$, $\lim_{k\to\infty} M_{k,\omega}(f)=f(z)$. Proof of (1). We have $M_{1,\omega}(P_n)=0$ by (a), hence $M_{1,\omega}(f)=0$ near $z_0$ by (d). Hence by (b) $M_{1,\omega}\{M_{n+1,\omega}(f)\}=0$ and $M_{n+1,\omega}(f)$, as a solution of the linear differential equation with constant coefficients $M_{1,\omega}(y)=0$, is an exponential polynomial and so by (d) $f(z)$ is the limit of a sequence of entire functions.

Proof of (2). If the domain of regularity of $f(z)$ is not convex we can find a singular point $\xi$ of $f(z)$ through which an arc of a circle $C$ can be drawn lying in $D$ except for $\xi$. We represent $f(z)$ as $f_1(z)+f_2(z)$, where $f_j(z)$ is the Cauchy integral of $f(z)$ along an arc $L_j$ such that $L_1+L_2$ is a simple contour on which $f(z)$ is regular and $L_2$ separates $L_1$ from $\xi$. Then $f_1(z)$ is regular at $\xi$ and so $f_2(z)$ is singular at $\xi$. We can represent $f_2(z)$ as $\int_0^{\infty e^{i\varphi}}\Phi(t)e^{-zt}dt$, where $\Phi(t)$ is entire, of order 1 and finite type $\rho$, the radius of a circle $C_1$ passing through $\xi$ and containing $C$, on which $f_2(z)$ has no singularity except $\xi$. It follows easily that $\xi$ is a singular point of $M_{1,\omega}(f_2)$; in fact $M_{1,\omega}(f_2)=\int_0^{\infty e^{i\varphi}}\Phi(t)L_{1,\omega}(t)e^{-zt}dt$. Since $\sigma=0$, $L_{1,\omega}(t)$ is of growth at most order 1, minimum type. Hence $M_{1,\omega}(f_2)$ has a singular point on $C_1$, and this can only be $\xi$. Since $M_{1,\omega}(f_1)+M_{1,\omega}(f_2)=M_{1,\omega}(f)=0$ and $M_{1,\omega}(f_1)$ is regular at $\xi$ we obtain a contradiction. Proof of (3). We have $M_{n+1,\omega}(f)=\lim_{k\to\infty} M_{n+1,\omega}(P_k)=\lim_{k\to\infty}\sum_{j=1}^{n}a_{k,j}L_{n+1,\omega}(\lambda_j)e^{\lambda_j z}$ and the same relation with primed letters, whence $a_{k,j}$ and $a'_{k,j}$ both approach the same limit as $k\to\infty$.

*R. P. Boas, Jr.* (Providence, R. I.).

**Dvoretzky, Aryeh. On the theorem of Jentzsch.** Proc. Nat. Acad. Sci. U. S. A. **35**, 246–252 (1949).

The author states a number of theorems concerning the location of the zeros of the polynomial sections $s_n(z)$ of the Taylor series of an analytic function $f(z)=\sum a_n z^n$ and the manner in which they accumulate near the circle of convergence. If $\Delta_n$ represents the distance of the point $z_0$ on the circle of convergence from the set of zeros of $s_n(z)$, then the well-known result of Jentzsch can be expressed as $\liminf_{n\to\infty}\Delta_n=0$. The manner in which certain subsequences of the $\Delta_n$ approach zero, however, can be described more precisely under various assumptions on the function $f(z)$. For example, if $f(z)$ is regular in $|z|<1$ and $\log f(z)=o(\log|1-z|)$ as $z$ approaches 1 in an angle, and if

$$-\infty<\limsup_{n\to\infty}(\log|a_n|/\log n)<\infty,$$

then there exists a subsequence of the $\Delta_n$ (here $z_0=1$) with order of magnitude not exceeding $n^{-1}\log n$. If, moreover, $f(z)$ is regular and not equal to zero at $z=1$ and if $0<\limsup_{n\to\infty}|a_n|<\infty$, then $\liminf_{n\to\infty} n\Delta_n<\infty$.

*E. N. Nilson* (Hartford, Conn.).

**Heuser, P. Zur Theorie der Tschebyscheffschen Polynome.** Math. Z. **51**, 574–585 (1949).

Let $C$ be a Jordan curve in the $x$-plane, $C_r$ the level curves corresponding to the concentric circles $|t|=r$ in the conformal mapping $x=\psi(t)=t+a_0+a_1t^{-1}+\cdots$, $|t|>r_0$, onto the exterior of $C$, $r>r_0$. Let $a_0=0$. We denote by $f_n(r;x)$ the Chebyshev polynomial of degree $n$ associated with $C_r$ (i.e., the polynomial with leading term $x^n$ and with a minimum of the maximum modulus on $C_r$). The author proves that $f_n(r;x)$ is continuous in $r$ and tends towards the Faber polynomial of degree $n$ associated with $C$ (or $C_r$) as $r\to\infty$.

*G. Szegö* (Stanford University, Calif.).

**Behnke, Heinrich, und Stein, Karl. Entwicklung analytischer Funktionen auf Riemannschen Flächen.** Math. Ann. **120**, 430–461 (1949).

A generalization of Runge's theorem in the theory of analytic functions of a complex variable to the case of domains on Riemann surfaces is given. Let the Riemann surface $R$ be a part of an open Riemann surface $R^*$. The authors show that a necessary and sufficient condition, in order that regular functions on $R$ may be approximated uniformly in $R$ by a set of functions which are regular in $R^*$, is that $R$ is simply connected relative to $R^*$. The surface $R$ is said to be simply connected relative to $R^*$ if every finite system of closed Jordan curves in $R$ which bounds a region in $R^*$ also bounds a region in $R$.

In order to prove the main theorem, the authors introduce a generalized Cauchy formula over the boundary $C$ of a polygon $G$ on $R$: $f(z)=(2\pi i)^{-1}\int_C f(\zeta)A(\zeta,z)d\zeta$, where $A$ is a single-valued function of the variables $p_1(\zeta)$ and $p_2(z)$ on the product domain $G_\zeta\times G_z$, which is meromorphic in $G_\zeta\times G_z$ and which has the property that the differential $A(\zeta,z)d\zeta$ remains finite for $p_1(\zeta)\neq p_2(z)$ and has a pole with a residue 1 for $p_1(\zeta)=p_2(z)$. The existence of such transcendental functions is proved by showing that every $G$ can be embedded in an algebraic surface. Then $A(\zeta,z)$ can be obtained as a sum of Abelian integrals of the second kind to which suitably chosen integrals of the second kind with poles outside $G$ are added. Using the above representation, the authors then prove, by shifting the poles in the usual way, the analogue of Runge's theorem.

The second part of the paper is devoted to the following question. What properties must be possessed by the Riemann surface $R^*$, which includes a Riemann surface $R$, in order that every single-valued regular function in $R$ can be approximated by functions which are single-valued and meromorphic in $R^*$, uniformly in $R$? The authors show that every single-valued regular function in $R$ can always be uniformly approximated by a function which is meromorphic in $R^*$ and which possess only finitely many poles, which lie on the boundary of $R$.

An elementary function $A(\zeta,z)$ for the entire Riemann surface $R^*$ (and not only for the subdomain $R$) is used in the above proof. The authors construct on $R^*$ a suitable integral $N(\zeta,z)$ of the third kind which depends analytically upon both variables and whose modules of periodicity possess certain properties. (They use the fact that on an open Riemann surface there always exists an integral of the first kind with the prescribed modules of periodicity). The function $A(\zeta,z)$ is obtained by differentiating $N(\zeta,z)$.

*S. Bergman* (Cambridge, Mass.).

**Nehari, Zeev. Sur un théorème de M. Montel.** C. R. Acad. Sci. Paris **228**, 1325–1327 (1949).

In connection with a theorem of Montel [Bull. Soc. Math. France **58**, 105–126 (1930)] the author proves the following statement. Let $f(z)$ be analytic and $|f(z)|<1$ for $|z|<1$, real for real $z$, $f(\beta)=f(-\beta)=0$, $0<\beta<1$, and $f(z)\neq 0$ for $z\neq\pm\beta$. Then $f'(z)$ has at least one zero in the interval $-\lambda<z<\lambda$ where the best $\lambda$ can be determined explicitly in terms of $\beta$ and $\alpha=f(0)$. *L. Ahlfors* (Cambridge, Mass.).

**Nehari, Zeev. The Schwarzian derivative and schlicht functions.** Bull. Amer. Math. Soc. **55**, 545–551 (1949).

If $g(z)=z^{-1}+a_0+a_1z+\cdots$ is schlicht in the unit circle, then $|a_1|\leqq 1$. If $w=f(z)$ is an analytic function, schlicht for

$|z|<1$, and

$$g(z)=\frac{f'(x)(1-|x|^2)}{f((x+z)/(1+\bar{x}z))-f(x)},$$

then $a_1=\frac{1}{6}(1-x^2)^2\{w,x\}$, where $\{w,x\}$ is the Schwarzian derivative of $w$ with regard to $x$. Thus $|\{w,z\}|\leqq 6(1-|z|^2)^{-2}$. The author shows that if the number 6 in this inequality is replaced by 2, the inequality is then sufficient for $w=f(z)$ to be schlicht for $|z|<1$. Since $\{w,z\}=2p(z)$ has its general solution of the form $y_1/y_2$, where $y_1$ and $y_2$ are linearly independent solutions of (A) $y''+p(z)y=0$, the author reduces the problem to one of showing that (A) has no solution which can vanish in $|z|<1$ more than once, provided $p(z)$ is bounded appropriately.

The author also shows that if, for $|z|<1$, $|\{w,z\}|\leqq\frac{1}{2}\pi^2$, then $w=f(z)$ is schlicht in the unit circle; the function $e^{\pi i(1+\epsilon)z}$, $\epsilon>0$, shows that the constant $\frac{1}{2}\pi^2$ is the best possible. A corollary is that the radius of univalence of $w=f(z)$ is at least equal to the smallest positive root of the equation $2\rho^2\max_{|z|=\rho}|\{w,z\}|=\pi^2$. *M. S. Robertson.*

**Hille, Einar. Remarks on a paper by Zeev Nehari.** Bull. Amer. Math. Soc. **55**, 552–553 (1949).

This note shows that the constant 2 in the paper reviewed above is the best possible one. For, the function

$$f(z)=\{(1-z)/(1+z)\}^{\epsilon i},$$

$\epsilon$ real, $f(0)=1$, takes on the value one infinitely often for $|z|<1$, and $|\{f(z),z\}|\leqq 2(1+\epsilon^2)(1-|z|^2)^{-2}$ with equality for real values of $z$. The author arrives at the function $f(z)$ by considering the equation $\{f(z),z\}=2a(1-z^2)^{-2}$, where $a$ is an arbitrary parameter which is associated with the differential equation $d^2y/dz^2+a(1-z^2)^{-2}y=0$. The region $U$ of the $a$-plane, giving rise to a family of schlicht functions $f(z)$, is completely determined. *M. S. Robertson.*

**Mullender, P. On some conformal mappings.** Simon Stevin **26**, 136–142 (1949). (Dutch)

A series $w(z)=z+\sum_2^\infty a_nz^n$ is called a Remak series if $\sum_2^\infty n|a_n|\leqq 1$ and $w'(1)=0$. For a Remak series $a_n\leqq 0$, $n=2,3,\cdots$. A function defined by such a series is said to have a cusp at $w(1)$ if $\lim_{\theta\to 0}\arg\{w(e^{i\theta})-w(1)\}=0$. Remak proved [Mathematica, Zutphen. B **11**, 175–192; **12**, 43–49 (1943); these Rev. **8**, 22] that if a Remak series has only a finite number of nonzero terms then the function has a cusp at $w(1)$, and he gave an example of a Remak series which has an infinite number of nonzero terms, but the function has no cusp. The author gives a simpler example of the same thing. The author also gives an example of a Remak series with an infinite number of nonzero terms for which the function does have a cusp. [In the first paragraph the author misquotes Remak's definition of the Hurwitz class of functions, and one of Remak's theorems; thus giving a false necessary condition that a function be univalent. These two errors have no effect on the validity of the author's results.] *A. W. Goodman* (Lexington, Ky.).

**Speiser, Andrea. Sulle superficie Riemanniane.** Rend. Sem. Mat. Fis. Milano **18** (1947), 91–92 (1948).

Report of a lecture.

***Čebotarëv, N. G. Teoriya Algebraičeskih Funkcil. [Theory of Algebraic Functions].** OGIZ, Moscow-Leningrad, 1948. 396 pp.

The author points out in the introduction that the Russian books on the theory of algebraic functions (in one variable) are few in number and are out of date. The present book is intended to fill this gap in the Russian mathematical literature. Roughly one half of the book is given to the arithmetic theory [Dedekind-Weber]. The other half presents the algebraic functions in the complex domain. The geometry of the subject matter is not included in the exposition. On the other hand, the author does not address himself exclusively to algebraists. For this reason, the arithmetic part of the book deals primarily with ground fields which are algebraically closed, and in the proofs it is (as a rule) tacitly assumed that the characteristic of the field is zero. There are frequent digressions to general ground fields, but for the case of a nonzero characteristic these digressions are sketchy and incomplete. The classical part of the book deals with such basic topics as Abelian integrals, Abel's theorem, the inversion problem and the Riemann $\vartheta$-function. The last chapter gives a brief account [8 pages] of rational points on an algebraic curve and on the $Z$-function in algebraic function fields over finite ground fields.

As a warning to the Russian reader, attention must be called to a somewhat confused state of affairs in the all-important chapter II in which the basic results which pertain to the notion of a "place" are developed. The crucial existence theorem ("there exists at least one place over every point of the curve $f(x,y)=0$") is not only not proved but not even mentioned. The terms "place" (or "prime divisor") and "prime ideal" are used interchangeably, but it is not proved that a prime ideal (in the ring of integral functions of $x$) determines a place. The proof of the basic theorem "number of zeros = number of poles" is incomplete, partly because of the reasons just mentioned. It may also be mentioned that the proof of the existence of a derivative [pp. 100–101] is incomplete and that the very definition of a derivative on p. 100 is not the right one in the case of nonzero characteristic.

The following is a summary of the contents of the various chapters. I. Field theory (algebraic field extensions, norms, traces, the discriminant, theorem of Lüroth). II. Fields of algebraic functions (rings of integral functions, normal integral bases, divisors, the divisor representation of a function, the case of a ground field which is not algebraically closed). III. Dimension of (divisor) classes (derivatives, differentials, the genus, ground field extensions). IV. Theorem of Riemann-Roch and applications (complementary normal bases, the gap theorem of Weierstrass-Noether, the theorem of Clifford, the proof of the Riemann-Roch theorem for arbitrary ground fields). V. Structure of algebraic function fields (birational transformations, singular points, resolution of the singularities by Kronecker's method, moduli, birational transformations of an algebraic curve into itself). VI. Applications of the theory of analytic functions (Newton's diagram, explicit determination of integral bases). VII. The Riemann surface. VIII. Abelian integrals. IX. Classical problems in the theory of algebraic functions (the Riemann $\vartheta$-function, the inversion problem of Jacobi, hyperelliptic curves, the principle of correspondence, reducible Abelian integrals, the problem of uniformization). X. Modern problems in the theory of algebraic functions. At the end of the book there is an extensive bibliography, together with directions to guide the reader in the literature of the subject.

*O. Zariski* (Cambridge, Mass.).

**af Hällström, Gunnar. Rationale Funktionen mit Automorphiefunktionen gleicher Vieldeutigkeit.** Acta Acad. Aboensis 16, no. 2, 10 pp. (1948).

**af Hällström, Gunnar. Über die Automorphiefunktionen meromorpher Funktionen.** Acta Acad. Aboensis 16, no. 4, 28 pp. (1949).

These papers are a continuation of the author's previous work [same Acta 15, nos. 6, 8 (1946); C. R. Dixième Congrès Math. Scandinaves 1946, pp. 97–107 (1947); these Rev. 9, 233] on analytic functions $f(z)$ which possess substitutions $f[S(z)]=f(z)$. Generalizing the term "automorphic functions," the author speaks about "half-automorphic" functions $f(z)$ if the $S(z)$ are two-valued algebraic functions, and similarly in higher cases. In the first paper, examples of "part-automorphic" rational functions of the orders 2, 3, 4 and 6 are constructed. In the second paper, a detailed discussion is given of the case in which $f(z)$ is a meromorphic function. In addition to rational or algebraic functions $S(z)$, there now also appear various possible types of transcendental functions $S(z)$. Examples are given.

*Z. Nehari* (St. Louis, Mo.).

**Petersson, Hans. Über die Transformationsfaktoren der relativen Invarianten linearer Substitutionsgruppen.** Monatsh. Math. 53, 17–41 (1949).

The author seeks to generalize the notion of automorphic forms by replacing the usual transformation equation $f(S\tau)=v(S)(c\tau+d)^r f(\tau)$ by (1) $f(S\tau)=R(\tau, S)f(\tau)$, where $S\tau=(a\tau+b)/(c\tau+d)$, $\det S\neq 0$, and investigates the problem in the following form: to find the set of functions ("transformation factors") $R(\tau, S)$, which are rational in $\tau, a, b, c, d$ and which, except for manifolds of lower dimension, satisfy the functional equation

$$R(S\tau, M)R(\tau, S)=\sigma_R(M, S)R(\tau, MS) \tag{2}$$

for all complex $\tau$ and all two-rowed nonsingular matrices $M, S$, where $\sigma_R(M, S)$ is a complex number independent of $\tau$. (When $\sigma_R\equiv 1$, (2) is a consequence of (1); the factor $\sigma_R(M, S)$ is inserted for convenience in connection with certain non-integral powers discussed below.) Under certain assumptions on $R(\tau, S)$ the author finds that $R(\tau, S)$ must have the form ("rational transformation factor")

$$R(\tau, S)=H(S)(c\tau+d)^r Q(S\tau)/Q(\tau), \tag{3}$$

where $\sigma_R(M, S)=H(M)H(S)/H(MS)$, $r$ is integral, and $Q$ and $H$ are rational functions of $\tau$ and of $a, b, c, d$, respectively, neither vanishing identically. When $S$ runs over a discrete group $\Gamma$ of substitutions, $Q(\tau)$ may be taken as a finite product of linear factors raised (under certain conditions) to complex powers and $H(S)$ is replaced by the usual "multiplier" $v(S)$ ("quasi-rational transformation factors"). If $\Gamma$ is a limit-circle group on the upper half-plane, any solution $f(\tau)$ of (1) with rational or quasi-rational $R(\tau, S)$, and which is analytic in the upper half-plane except possibly for poles in the local variable, is equal to a usual automorphic form on $\Gamma$ times $Q(\tau)$. The author concludes by extending these results to hyperabelian transformation groups in $n$ complex variables.

*J. Lehner* (Philadelphia, Pa.).

**Položiĭ, G. N. Singular points and calculation of $p$-analytic functions of a complex variable.** Doklady Akad. Nauk SSSR (N.S.) 60, 769–772 (1948). (Russian)

The author announces a classification of isolated singularities of single-valued $p$-analytic functions [Položiĭ, Doklady Akad. Nauk SSSR (N.S.) 58, 1275–1278 (1947); these Rev. 9, 507], i.e., of functions $f=u+iv$ with $u_x=p(x, y)v_y$, $u_y=-p(x, y)v_x$. The "characteristic" $p(x, y)$ is to possess partial derivatives which are assumed either ($\alpha$) to be continuous, or ($\beta$) to satisfy Hölder conditions. If ($\alpha$) is assumed and $f(z)$ is single-valued and bounded in the neighborhood of $z_0$, then $\lim_{z\to z_0} f(z)$ exists and is finite (removable singularity). Under hypothesis ($\beta$) it follows that $f(z)$ is $p$-analytic at $z_0$. Say that $f(z)$ has a pole at $z_0$ if $f(z)\to\infty$ for $z\to z_0$. In this case, if ($\beta$) holds and $f(z)=O(|z-z_0|^{\rho})$ for some positive $\rho$, then there exist a positive integer $n$ (order of the pole), and positive constants $A, B$ such that $A<|f(z)(z-z_0)^n|<B$ and

$$(p^{\frac{1}{2}}u+ip^{-\frac{1}{2}}v)(z-z_0)^n$$

approaches a finite limit $\lambda$ as $z\to z_0$. If $n=1$ the number $\{p(z_0)\}^{\frac{1}{2}}\lambda=c_{-1}$ is called the residue of $f$. If ($\beta$) holds and $f$ and $g$ have first order poles at $z_0$ with equal residues, then $f-g$ is $p$-analytic at $z_0$. Under hypothesis ($\beta$) Liouville's theorem holds for $p$-analytic functions. Under assumption ($\alpha$) Weierstrass' theorem holds; i.e., if $f(z)$ is $p$-analytic in the neighborhood of $z_0$ and $f(z)$ has no finite or infinite limit as $z\to z_0$ (essential singularity), then $f(z)$ comes arbitrarily close to any complex number in every neighborhood of $z_0$. For functions having only simple poles in a domain the author announces (under hypothesis ($\beta$)) a theorem analogous to the residue theorem.

*L. Bers.*

**Esteban Carrasco, Luis. Solution of a problem on the derived circular congruence of a polygenic function.** Revista Mat. Hisp.-Amer. (4) 9, 10–12 (1949). (Spanish)

Let $w=w(z)=u(x, y)+iv(x, y)$ define a polygenic function of the complex variable $z=x+iy$ over a region $E$ of the $z$-plane. At each point $z$ of $E$, the derivative $\gamma=\alpha+i\beta=dw/dz$ is represented by a circle, called the Kasner circle. Thus the $\infty^2$ points of $E$ are mapped into a set of $\infty^2$ circles of the $\gamma$-plane. This correspondence is termed the derivative circular congruence representation of the polygenic function. The analytic conditions for a congruence representation of circles to be a derivative congruence are obtained in papers by Kasner [Bull. Amer. Math. Soc. 34, 561–565 (1928)] and Kasner and De Cicco [Amer. J. Math. 61, 995–1003 (1939); these Rev. 1, 48, 400]. As the point $z$ in $E$ describes a curve $C$, the corresponding derivative circles form a family $F$. The author discusses the following two problems proposed by De Cicco. (1) For a given polygenic function, to find the curves $C$ such that the corresponding family $F$ of $\infty^1$ circles forms an osculating set. (2) To obtain the conditions on $w=u+iv$ such that the preceding condition is valid for every curve $C$ of $E$. The curves $C$ of problem (1) form a net of $2\infty^1$ curves. The polygenic functions $w=u+iv$ of problem (2) obey a system of three partial differential equations in $u$ and $v$, each of the second order.

*J. De Cicco* (Chicago, Ill.).

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## *Theory of Series*

**Jarden, Dov, and Motzkin, Theodor. The product of sequences with a common linear recursion formula of order 2.** Riveon Lematematika 3, 25–27, 38 (1949). (Hebrew. English summary)

A recurring sequence $(R_n)$ of order $r$ with the alternating scale $a_0, \cdots, a_r$, where $a_0, \cdots, a_r$ are arbitrary complex numbers with $a_0a_r\neq 0$, is a sequence for which

$$\sum_{i=0}^{r}(-1)^i a_i R_{n+i}=0, \quad n=\cdots, -1, 0, 1, \cdots.$$

We consider $k-1$ ($k>1$) recurring sequences $(W_n^{(i)})$ ($i=1, \cdots, k-1$) of order 2 with the common alternating scale $a, b, c$. The fundamental recurring sequence $(U_n)$ with the alternating scale $a, b, c$ is defined by $U_0=0$, $U_1=1$. We call $\binom{k}{i}_U = U_kU_{k-1}\cdots U_{k-i+1}/U_1U_2\cdots U_i$, and $\binom{k}{0}_U=1$, a generalized binomial coefficient formed from the sequence $(U_n)$.

Theorem 1. The sequence $(P_n=\prod_{i=1}^{k-1}W_n^{(i)})$ whose $n$th term is the product of the $n$th terms of $k-1$ ($k>1$) recurring sequences $(W_n^{(1)}), \cdots, (W_n^{(k-1)})$ with the common alternating scale $a, b, c$ ($ac\neq 0$), is a recurring sequence of order $k$ with the alternating scale

$$s_i=\left(\frac{a}{c}\right)^{\binom{k-1}{2}}\binom{k}{i}_U, \qquad i=0, \cdots, k.$$

Theorem 2. For the fundamental recurring sequence $(U_n)$ with the alternating scale $a, b, 1$, where $a$ and $b$ are integers, every generalized binomial coefficient $\binom{k}{i}_U$ is an integer.

*From the authors' summary.*

**Tenca, Luigi. Ricerche sui determinanti di somme.** Pont. Acad. Sci. Acta **11**, 149–156 (1947).

From the succession of real numbers $\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots$ successions of higher order are defined by $a_s^{(1)}=a_s+a_{s+1}$, $a_s^{(i+1)}=a_s^{(i)}+a_{s+1}^{(i)}$, $s=\cdots, -2, -1, 0, 1, 2, \cdots$. A succession is called a progression of sums of order $n$ and radius $d$, a real constant different from zero, provided $a_s^{(n)}=(-1)^s d$ for each integer $s$. The paper investigates some elementary properties of such progressions. *L. M. Blumenthal.*

**Wright, E. M. The Taylor coefficients of integral functions.** J. London Math. Soc. **24**, 40–43 (1949).

The author has deduced [Trans. Amer. Math. Soc. **64**, 409–438 (1948); these Rev. **10**, 441] the asymptotic behaviour of the coefficients $c_\lambda(n)$ of the integral function $f(x+\lambda)=\sum_{n=0}^{\infty}c_\lambda(n)x^n$ from that of the coefficients $c_0(n)$. Starting from the formula

$$c_\lambda(n)=\sum_{s=0}^{\infty}\binom{n+s}{s}c_0(n+s)\lambda^s,$$

he obtains here by a direct method a special case of the following theorem. If $c_0(n)\sim n^\beta e^{\psi}(e/\kappa n)^{\kappa n}$, where $\Re\kappa\geqq 1$ and $\psi$ is a sum of terms of the form $an^b$ ($0\leqq\Re b<1$), then $c_\lambda(n)\sim e^\chi c_0(n)$, where $\chi=0$ if $\Re\kappa>1$ and $\chi=\lambda\kappa^{-\kappa}n^{1-\kappa}$ if $\Re\kappa=1$. *J. G. van der Corput* (Amsterdam).

**Pettineo, Benedetto. Sulle serie numeriche $\sum u_n f(n)$, dove la successione $u_n$ è monotona e la funzione $f(x)$ è periodica.** Boll. Un. Mat. Ital. (3) **4**, 61–63 (1949).

Cf. the author's paper [Ann. Mat. Pura Appl. (4) **26**, 119–140 (1947); these Rev. **10**, 112].

**Rajagopal, C. T. Some theorems concerning Riesz's first mean.** Acad. Serbe Sci. Publ. Inst. Math. **1**, 11–20 (1947).

Let $\bar{s}$, $\underline{s}$, $\bar{\sigma}$, $\underline{\sigma}$ be the superior and inferior limits of the partial sums and $(R, \lambda, 1)$ means, respectively, of a series $\sum a_n$. Write $A(t)=\sum_{\lambda_n\leqq t}a_n$, where $0<\lambda_0<\lambda_1<\cdots$, $\lambda_n\to\infty$, and let

$$\limsup_{x\to\infty}\max_{x\leqq x'\leqq tx}\{A(x')-A(x)\}\leqq W(t),$$

$$\liminf_{x\to\infty}\min_{x\leqq x'\leqq tx}\{A(x')-A(x)\}\geqq w(t).$$

The following results (in increasing order of generality) are known. (1) If $\bar{\sigma}=\underline{\sigma}$ and $W(t)\to 0$ as $t\to+0$, then $\bar{s}=\underline{s}$, (2) if $0<\theta<1<\lambda$, then

$$(\lambda-1)\underline{s}\geqq\lambda\underline{\sigma}-\bar{\sigma}-\int_1^\lambda W(t)dt,$$

$$(1-\theta)\bar{s}\leqq\bar{\sigma}-\theta\underline{\sigma}+\int_\theta^1 W(t^{-1})dt,$$

and

$$(3)\qquad (\lambda-\theta)\underline{s}\geqq\lambda\underline{\sigma}-\theta\bar{\sigma}-\int_1^\lambda W(t)dt-\int_\theta^1 w(t^{-1})dt,$$

$$(\lambda-\theta)\bar{s}\leqq\lambda\bar{\sigma}-\theta\underline{\sigma}+\int_\theta^1 W(t^{-1})dt+\int_1^\lambda w(t)dt.$$

[(1) is included in a theorem of O. Szász, S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1929**, 325–340; (2) and (3) were given by J. Karamata, Proc. London Math. Soc. (2) **43**, 20–25 (1937); earlier results were obtained by L. Fejér, C. Nagy, M. Fekete and C. E. Winn; extensions to higher Riesz means have been given by S. Minakshisundaram, J. Indian Math. Soc. (N.S.) **3**, 127–130 (1938), and C. T. Rajagopal, J. London Math. Soc. **21** (1946), 275–282 (1947); these Rev. **9**, 86.] Here the author obtains similar results in which $a_n\leqq(\lambda_{n+1}-\lambda_n)F(\lambda_{n+1})$, where (i) $\lambda_n\sim\lambda_{n+1}$, (ii) $F(x)$ decreases, (iii) $\int^x F(\xi)d\xi$ satisfies conditions like those on $A(t)$ above. He also gives various known results as corollaries. The reviewer observes that the author's results may be deduced from (1)–(3) above.

*L. S. Bosanquet* (London).

**Basu, S. K. Note on some theorems on the Hölder and Cesàro means.** Bull. Calcutta Math. Soc. **40**, 129–134 (1948).

Let $A$ and $B$ be equivalent Hurwitz-Silverman-Hausdorff transformations generated by moment sequences of non-vanishing real numbers, and let $A_n$ and $B_n$ be the corresponding transforms of a real sequence $s_n$. If $r>0$ and $A_n=O(n^r)$ then $B_n=O(n^r)$ and conversely. The two propositions, (1) if $B_n\geqq 0$ when $n=0, 1, 2, \cdots$, then $A_n\geqq 0$ when $n=0, 1, 2, \cdots$ and (2) if $B_n=O_L(n^r)$ for a given $r>0$ then $A_n=O_L(n^r)$, are equivalent. Related results and applications to the Hölder and Cesàro transformations are given.

*R. P. Agnew* (Ithaca, N. Y.).

**Vermes, P. Series to series transformations and analytic continuation by matrix methods.** Amer. J. Math. **71**, 541–562 (1949).

The author considers series-to-series and series-to-sequence transformations by matrices, and establishes certain fundamental relations concerning the corresponding matrices and their matrix products. He then gives the motivation for dealing with the special matrices $A(p)=(a_{nk})$, where $a_{nk}=C_{kn}p^n(1-p)^{k-n}$ ($p$ is a complex parameter, fixed for each matrix; $C_{kn}$ is a binomial coefficient, equal to zero if $k<n$). These matrices represent series-to-series transformations; the associated sequence-to-sequence transformations are represented by matrices of the form $F(p)=pA(p)$. Matrix products satisfy the rule $A(p)A(q)=A(pq)$. Adjunction of elements to a series summable by $A(p)$ is always permissible; omission of elements is permissible provided $p=1$ or $|1-1/p|>1$. The author describes in detail the regions in which the transformations $A(p)$ sum a Taylor series to the analytic continuation of the function represented by the series. He discusses the relations between the matrices $A(p)$ and $F(p)$ and the Euler matrices which are obtained by transposing them. He exhibits a one-parameter

family of matrices that serve to sum Laurent series in certain regions inside the circle of convergence.

*G. Piranian* (Ann Arbor, Mich.).

**Schurr, Zvi. On absolute regularity of linear transformations.** Riveon Lematematika **2**, 30–33 (1948). (Hebrew)

This is a continuation of a paper by the same title in the same vol., 12–17 (1947); these Rev. **9**, 579.

*A. Dvoretzky* (Princeton, N. J.).

**Dieulefait, Carlos E. On the inversion of limits and analytic continuation.** An. Soc. Ci. Argentina **146**, 406–416 (1948). (Spanish)

Using the formal identity

$$\sum x^n=(1+x+x^2+\cdots+x^{k-1})\sum(x^k)^n,$$

and applying the Borel integral method of summability to the series $\sum(x^k)^n$ for fixed positive integer values of $k$, the author obtains methods for evaluating $\sum x^n$ to $1/(1-x)$. If $x_0$ lies in the Mittag-Leffler star of $1/(1-x)$, the method is effective for large values of $k$. By use of the Cauchy integral formula and other formulas, the ideas are extended to obtain methods for evaluation of other series.

*R. P. Agnew* (Ithaca, N. Y.).

**Tricomi, Francesco. Sugli zeri delle funzioni di cui si conosce una rappresentazione asintotica.** Ann. Mat. Pura Appl. (4) **26**, 283–300 (1947).

The main theorem is as follows. If a continuous function $f(x)$ can be represented asymptotically (uniformly with respect to $x$): $f(x)=\sum_{k=0}^{n}G_k(x)\mu^k+O(\mu^{n+1})$, $\mu\to 0$, where $G_k(x)$ is a function which is differentiable at least $n-k+1$ times in the neighborhood of a simple zero $x_0$ of the function $g_0(x)$, the equation $f(x)=0$ has at least one zero $x_0^*$ in the interval $(x_0-\epsilon, x_0+\epsilon)$, $\epsilon>0$ arbitrarily small, for $|\mu|<\delta$ ($\delta$ suitably chosen), and

$$x_0^*=x_0+\sum_{k=1}^{n}W_{k-1}\mu^k+O(\mu^{n+1}), \tag{1}$$

where the quantities $W_k$ are well-determinated rational functions of $G_{km}=(1/m!)G_k^{(m)}(x_0)$. The first two are: $W_0=-G_{10}/G_{01}$; $W_1=-(G_{10}^2G_{02}-G_{10}G_{01}G_{11}+G_{01}^2G_{20})/G_{01}^3$. Application of (1) is made to the Bessel functions, the confluent hypergeometric functions (especially the Laguerre polynomials) and to the functions of Legendre. In the latter case the following simple approximate expression

$$x_r^*=\cos\left(\frac{4r-1}{4n+2}\pi\right)\left\{1-\frac{1}{2(2n+1)(2n-3)}\right\}+O(n^{-1}),$$

$$r=1, 2, 3\cdots n,$$

is obtained for the zeros of $P_n(\cos\theta)$. *S. C. van Veen.*

**Edrei, Albert. Sur des formules d'inversion pour les transformées de Stieltjes et certains théorèmes taubériens.** C. R. Acad. Sci. Paris **228**, 1365–1367 (1949).

The author states two theorems of a Tauberian nature concerning the transform $f(z)=\int_0^\infty(t+z)^{-1}d\psi(t)$. The first of these generalizes a Tauberian theorem of Hardy and Littlewood [Proc. London Math. Soc. (2) **30**, 23–37 (1929), theorem 4, p. 25]. The second of them is analogous to a theorem of Valiron [Ann. Fac. Sci. Univ. Toulouse (3) **5**, 117–257 (1914), pp. 123–127] and states precise conditions under which one can conclude from a hypothesis of the form $\int_0^\infty(t+r)^{-1}\Phi(t)dt\sim g(r)$ $(r\to\infty)$ that $\Phi(r)\sim Cg(r)$, $r\to\infty$.

*H. Pollard* (Ithaca, N. Y.).

**Frank, Evelyn. Orthogonality properties of *C*-fractions.** Bull. Amer. Math. Soc. **55**, 384–390 (1949).

A *C*-fraction is a continued fraction of the form

$$1+K(a_nx^{n_p}/1), \tag{1}$$

where the $a_n$ are positive integers. It is shown that a weak orthogonality relation exists in the set of polynomials $\{B_n^*\}$. Here $B_p^*=x^{n_p}B_p(1/x)$, where $B_p(x)$ is the denominator of the $p$th approximant of (1) and $n_p=\max(s_p, t_p)$, where $s_p$ is the degree of $A_p(x)$ (the numerator of the $p$th approximant of (1)) and $t_p$ is the degree of $B_p(x)$. The orthogonality relation is such that in certain cases $B_n$ is orthogonal to itself for all $n>n_0$. [The author wishes to make the following changes. On p. 386, just before formula (2.7), replace "From the algorithm (2.4)" by "By equating coefficients of like powers of $z$ in the formula

$$P(z)B_p(z)-A_p(z)=(-1)^pa_1a_2\cdots a_{p+1}z^{n_1+n_2+\cdots+n_{p+1}}$$

(cf. p. 385), one finds that." A similar statement should be made on p. 387 just preceding formula (2.10).]

*W. J. Thron* (St. Louis, Mo.).

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## *Fourier Series and Generalizations, Integral Transforms*

**Schmetterer, Leopold. Zur Fourierentwicklung des Produktes zweier Funktionen.** Monatsh. Math. **53**, 53–62 (1949).

Let $f(x)$ and $g(x)$ be of period $2\pi$ and $L$-integrable, and suppose that the Fourier series of $f$ converges at $x_0$. The author discusses the conditions under which the convergence of the Fourier series of $fg$ at $x_0$ can be established. It is known that this is true when (i) $f$ is bounded and $g$ satisfies at $x_0$ the condition of Dini [Steinhaus, Bull. Intern. Acad. Cracovie Cl. Sci. Math. Nat. Sér. A. **1913**, 113–116], or (ii) $f$ is $L$ and $g$ has bounded difference quotients at $x_0$ [W. H. Young, Proc. London Math. Soc. (2) **18**, 141–162 (1919), in particular, p. 144]. It is first shown, by an example of the Fejér type, that, even if $f$ is continuous and $g$ is absolutely continuous, the convergence of the Fourier series of $f$ need not imply that of the Fourier series of $fg$. On the other hand, this is implied when $f$ is of bounded variation and $g$ is absolutely continuous, or vice versa. If $f$ is even and $L^p$, $p>1$, and $g$ is $L^{p'}$, then the Fourier series of $fg$ converges at $x=0$, provided that both the Fourier series of $f$ and $g$ converge at $x=0$, at least one of them absolutely. *W. W. Rogosinski* (Newcastle-upon-Tyne).

**Kawata, Tatsuo. The Lipschitz condition of a function and Fejer means of Fourier series.** Kōdai Math. Sem. Rep., no. 1, 1–4 (1949).

The well-known result of S. Bernstein [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 4°. (2) **4**, no. 1 (1912)] states that if $f(x)\in\mathrm{Lip}\,\alpha$ and $\sigma_n(x)$ is the $(C, 1)$ mean of the Fourier series of $f(x)$ then (A) $f(x)-\sigma_n(x)=O(n^{-1})$ uniformly in $x$ holds if $0<\alpha<1$ but not for $\alpha=1$. For $\alpha=1$, Zygmund [Bull. Amer. Math. Soc. **51**, 274–278 (1945); these Rev. **6**, 265] has shown that if $f(x)\in\mathrm{Lip}\,1$ and its Fourier series is of power series type then (A) holds. The present paper shows that if $f(x)\in\mathrm{Lip}\,1$, the conditions (B) $\bar{f}(x)\in\mathrm{Lip}\,1$ and (C) $\int_\eta^\pi\{f(x+t)+f(x-t)-2f(x)\}dt=O(\eta)$ uniformly in $x$ as $\eta\to 0$ are each equivalent to (A). [Condition (C) is misprinted in the paper.] It is also shown that

$f(x)\in \mathrm{Lip}\,1$ and (A) imply $\bar{f}(x)-\bar{\sigma}_n(x)=O(n^{-1})$, while (A) alone yields $\bar{f}(x)-\bar{\sigma}_n(x)=O(n^{-1}\log n)$. *P. Civin.*

**Lee, Kwok-Ping. Sur l'approximation des fonctions analytiques presque-périodiques.** Quart. J. Sci. Wu-Han Univ. 9, 17–31 (1948).

The author proves a series of theorems on a class of almost periodic analytic functions. We shall suppose that $0\leqq|\lambda_n|\leqq|\lambda_{n+1}|\to\infty$. The following two theorems are samples. If (i) $f(iy)\sim\sum a_{\lambda_n}e^{i\lambda_n y}$ is almost periodic, (ii) there exists a sequence $n_j$ such that $0<\lim_j(|\lambda_{n_j}|-|\lambda_{n_{j-1}}|)=\tau<\infty$ and (iii) there exist a non-decreasing function $\psi$ and a constant $h$ such that for all $j$ there exists a trigonometric polynomial $T_j=\sum_{|\lambda_n|\leqq|\lambda_{n_j}|}A_{\lambda_n}e^{i\lambda_n y}$ for which

$$|f(iy)-T_j(y)|\leqq he^{-|\lambda_{n_j}|\psi(|\lambda_{n_j}|)},$$

then $f(z)$ is an entire function, and

$$|f(\pm x+iy)|<e^{\varphi[\varphi(x+\epsilon)+k]},$$

where $\varphi$ denotes the inverse of $\psi$ and $k$ denotes a suitable constant. The following theorem is the inverse of the above theorem. If $f(z)\sim\sum a_{\lambda_n}e^{i\lambda_n z}$ is an entire almost periodic function such that $0<\lim(n/|\lambda_n|)=\rho<\infty$, and $\varphi$ is the smallest nondecreasing function such that $|f(\pm x+iy)|\leqq e^{y\varphi(x)}$, and if further $\lim_{x\to\infty}\varphi(x)=\infty$ and $\psi$ is the inverse of $\varphi$, then the best approximation $m_{\lambda_n}$ of $f(iy)$ by means of trigonometric polynomials of order $\lambda_n$ satisfies $m_{\lambda_n}<e^{-(1+\epsilon)|\lambda_n|\psi(|\lambda_n|+k)}$ for large enough $n$, arbitrary $\epsilon>0$ and suitable $k$. The proofs use careful estimates and a classical theorem of S. Bernstein on the derivative of a trigonometric polynomial.

*František Wolf* (Berkeley, Calif.).

**Alexits, Georges. Sur la convergence des séries orthonormales lacunaires.** Acta Univ. Szeged. Sect. Sci. Math. 13, 14–17 (1949).

The orthonormal series $\sum c_n\varphi_n(x)$ (over $[a,b]$) is said to be $\lambda_n$-lacunary if the number of $c_k\neq0$ with $2^n\leqq k<2^{n+1}$ is $O(\lambda_{2^n})$, where the $\lambda_n$ are nondecreasing positive integers. I. S. Gál [C. R. Acad. Sci. Paris 227, 1140–1142 (1948); these Rev. 10, 292] proved the following theorem. Suppose that the orthonormal series is $\lambda_n$-lacunary and that $\sum c_n^2(\log\lambda_n)^2<\infty$. If then the series is $(C,1)$-summable almost everywhere, it is convergent almost everywhere. The author gives a simple proof for this theorem and shows that it is, in a way, a best possible result.

*W. W. Rogosinski* (Newcastle-upon-Tyne).

**Alexits, Georges. Sur la convergence d'une classe de séries orthonormales.** Acta Univ. Szeged. Sect. Sci. Math. 13, 18–20 (1949).

It is shown that the orthonormal series $\sum c_n\varphi_n(x)$ (over $[a,b]$) converges almost everywhere if the $|c_n|$ decrease and $c_n=O(n^{-1}(\log n)^{-1-\epsilon})$ for some $\epsilon>0$.

*W. W. Rogosinski* (Newcastle-upon-Tyne).

**Weston, J. D. The cardinal series in Hilbert space.** Proc. Cambridge Philos. Soc. 45, 335–341 (1949).

The author starts from Hardy's observation [same Proc. 37, 331–348 (1941); these Rev. 3, 108] that the functions $v_r(t)=\pi^{-1}(t-r)^{-1}\sin\pi(t-r)$ $(v_r(r)=1)$, $r=0,\pm1,\pm2,\cdots$, are orthonormal on $(-\infty,\infty)$, and formulates a result of Hardy in the form that $\{v_r(t)\}$ is an orthonormal basis of the Hilbert space $H_2$ of functions of $L^2(-\infty,\infty)$ whose Fourier transforms vanish outside $(-\pi,\pi)$. He shows that if $f(x)$ is Riemann integrable in every finite interval and $O(|x|^{\alpha})$, $\alpha<-\frac{1}{2}$, at $\infty$, then $\sum_{-N}^{N}f(a+rw)v_r[(x-a)/w]$ converges in mean square to $f(x)$ as $N\to\infty$ and $w\to0$. He next considers some special transformations in $H_2$. Connections with Hardy's $m$-functions [Proc. London Math. Soc. (2) 7, 445–472 (1909)] and further properties of $H_2$ are given. *R. P. Boas, Jr.* (Providence, R. I.).

**Sbrana, F., e Fumi, F. Integrazione della equazione del calore unidimensionale per mezzo degli operatori funzionali in una variabile.** Atti Accad. Ligure 5, 27 pp. (1948).

The authors discuss a few boundary value problems of the partial differential equation $u_{xx}=au_t+bu$ ($a$, $b$ constants) by means of the operational calculus. They do not use the Laplace transform theory explicitly, but base their work on "Sommerfeld's theorem"

$$\Phi(y)=\lim_{\epsilon\to+0}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\omega^2}d\omega\int_{y_1}^{y_2}\Phi(\eta)e^{\omega(y-\eta)}d\eta,\qquad y_1<y<y_2.$$

The material is partly taken from a course of lectures given by the first named author. *A. Erdélyi.*

**Sbrana, Francesco. Sugli operatori funzionali multipli. I.** Atti Accad. Ligure 5, 14 pp. (1948).

The author develops an operational calculus in several variables by first representing a function in terms of exponential functions and then operating on the exponentials in the obvious manner. For the first step, he proves "Sommerfeld's theorem for functions of two variables" and an "exponential representation of zero." Let $A$ be a domain in the $(x,y)$-plane, $l$ be the boundary of $A$, $s$ be the length of arc on $l$ and $f(x,y)$ and $g(x,y)$ be continuous functions in $A$ and on $l$, respectively. Then

$$(1)\quad \lim_{\epsilon\to+0}\frac{1}{4\pi^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}d\omega_1d\omega_2$$
$$\times\iint_A f(\xi,\eta)\exp\{\epsilon(\omega_1^2+\omega_2^2)+\omega_1(x-\xi)+\omega_2(y-\eta)\}d\xi d\eta$$

is equal to $f(x,y)$ if $(x,y)$ is in $A$ and is equal to 0 if $(x,y)$ is outside $A$, and

$$(2)\quad \lim_{\epsilon\to+0}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}d\omega_1d\omega_2$$
$$\times\int_l g(\xi,\eta)\exp\{\epsilon(\omega_1^2+\omega_2^2)+\omega_1(x-\xi)+\omega_2(y-\eta)\}d\xi d\eta=0.$$

A few other related formulas are also developed. If $D_1=\partial/\partial x$, $D_2=\partial/\partial y$, and $E(D_1,D_2)$ is any differential operator, then $E(D_1,D_2)f(x,y)$ is defined by inserting the factor $E(\omega_1,\omega_2)$ in the integrands of both (1) and (2) and adding. Here $g$ is an arbitrary (continuous) function and provides the means of satisfying boundary conditions. *A. Erdélyi.*

**Sbrana, Francesco. Sugli operatori funzionali multipli. II.** Atti Accad. Ligure 5, 14 pp. (1948).

The operational calculus developed in the first part [cf. the preceding review] is used to obtain the solutions of the partial differential equation $(D_1^2+D_2^2)^n u=f(x,y)$ for $n=1$ (Poisson's equation), $n=2$ and also for general (integral) $n$.

*A. Erdélyi* (Pasadena, Calif.).

**Sbrana, Francesco. Sugli operatori funzionali multipli. III.** Atti Accad. Ligure **5**, 17 pp. (1948).

Sommerfeld's theorem and the exponential representation of zero developed in the first note [cf. the second preceding review] are extended to functions of three variables. The results are used to write down solutions of the equations

$$(\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2-K^2)u=f(x, y, z),$$
$$(\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2-V^{-2}\partial^2/\partial t^2)u=f(x, y, z, t)$$

and also of the system

$$\partial u/\partial x-\partial v/\partial y=f(x, y),\quad \partial u/\partial y+\partial v/\partial x=g(x, y).$$

*A. Erdélyi* (Pasadena, Calif.).

**Sbrana, Francesco. Sugli operatori funzionali multipli.** Boll. Un. Mat. Ital. (3) **4**, 34–40 (1949).

In the three papers reviewed above, the author has developed an operational calculus in two variables. He now uses this calculus to obtain a general solution of the partial differential equation $u_{xx}-u_y=0$, and indicates the solution of $u_{xxx}-u_y=0$. *A. Erdélyi* (Pasadena, Calif.).

**Udagawa, Masatomo. Some properties of asymptotic distributions.** Kōdai Math. Sem. Rep., no. **1**, 5–7, 17 (1949).

(1) If $x(t)$ has an asymptotic distribution function (a.d.f.), and if $f(x)$ is continuous, $f[x(t)]$ has an asymptotic distribution function. [For a discussion of a.d.f.'s, cf. Hartman, van Kampen and Wintner, Amer. J. Math. **61**, 477–486 (1939).] (2) If $x(t)$ and $y(t)$ are statistically independent and bounded on $(-\infty, \infty)$, and if $f$ and $g$ are continuous, $f[x(t)]$ and $g[y(t)]$ are statistically independent. [Cf. Kac and Steinhaus, Studia Math. **7**, 1–15 (1938), for this result under slightly different hypotheses.] (3) If $x_n(t)$ and $y_n(t)$ are statistically independent and if $x_n(t)$, $y_n(t)$ converge to $x(t)$, $y(t)$ in (relative) measure on $(-\infty, \infty)$, then $x(t)$ and $y(t)$ are statistically independent. (4) If $x_n(t)\to 0$, $y_n(t)\to y(t)$ in relative measure on $(-\infty, \infty)$ and if $x_n(t)$, $y_n(t)$, $x_n(t)+y_n(t)$ have a.d.f.'s then $x_n(t)+y_n(t)\to y(t)$ in relative measure. (5) If $x(t)$ has the unit a.d.f. and if $y(t)$ has an a.d.f. then $x(t)+y(t)$ has an a.d.f., that of $y(t)$. *J. L. Doob.*

**Hirschman, I. I., Jr. Approximation by non-dense sets of functions.** Ann. of Math. (2) **50**, 666–675 (1949).

Let $c_k$ be a set of positive numbers such that

$$\sum c_k/(1+c_k^2)<\infty.$$

If in addition (*) $c_k\to\infty$, $k\to\infty$, then the closure of $\{e^{-c_k x}\}$ in $L^2(0, \infty)$ consists of functions which admit analytic continuation into the right half-plane. [See, for example, L. Schwartz, Étude des sommes d'exponentielles réelles, Actual. Sci. Ind., no. 959, Hermann, Paris, 1943; these Rev. **7**, 294.] By use of the Hankel transformation of order zero, which converts $e^{-bx}$ into $b^{-1}e^{-x^2/b}$, the author applies Schwartz's work to show that if (*) is replaced by (**) $c_k\to 0$, then the closure now consists of entire functions belonging to $L^2$ on the positive real axis. A combination of the two results enables the author to remove condition (*) from the first one; the essential tool is the notion of complete disjointness of closed subspaces of $L^2$. By a further use of an integral transformation, the author shows that if $\sum(c_k^2/(1+c_k))<\infty$, then the $L^2$ closure of $\{1/(x+c_k)\}$ consists of functions analytic except on the negative real axis. *H. Pollard.*

## *Polynomials, Polynomial Approximations*

**Geller, L. B. Concerning aperiodic stability of linear systems.** Uspehi Matem. Nauk (N.S.) **4**, no. 2(30), 206–208 (1949). (Russian)

The author proves that if $p(x)=\sum_0^n c_k x^{n-k}$ has all roots real then

$$c_\nu^2\geqq(1+1/\nu)(1+1/(n-\nu))c_{\nu+1}c_{\nu-1},$$
$$\nu=1, 2, \cdots, n-1.$$

He points out that this result was proved by Euler and that the weakened version $c_\nu^2>c_{\nu+1}c_{\nu-1}$ was proved by de Gua. *A. W. Goodman* (Lexington, Ky.).

**Sz. Nagy, Gyula. Merkwürdige Punktgruppen bei allgemeinen Lemniskaten.** Acta Univ. Szeged. Sect. Sci. Math. **13**, 1–13 (1949).

An investigation is made of the location of certain point sets related to the lemniscate $L: |f(z)|=R$, where $f(z)=(z-z_1)(z-z_2)\cdots(z-z_n)$ and $R\geqq 0$. For any given $Z$, the zeros of the polynomial $f(z)-Z$ are said to form an inner, outer, or principal pole-set relative to $L$ according as $|Z|<R$, $|Z|>R$, or $|Z|=R$. The pole-sets corresponding to two different $Z$, $Z_1$ and $Z_2$, are said to be conjugate relative to $L$ if $|Z_1Z_2|=R^2$. The following theorems are among those established by elementary methods. (I) Any two inner pole-sets have the same number $k$ of points within each cycle $M$ of $L$. (II) Each principal pole-set has exactly $k$ distinct points on the boundary $m$ of $M$ and these points separate the points on $m$ belonging to any other principal pole-set. (III) The product of the distances of the points of a pole-set to a point $P$ of $L$ is $|Z|/R$ times the product of the distances of the points of the conjugate pole-set to $P$. (IV) If the cross-ratio $C=(Z_1, Z_2; Z_3, Z_4)$ of the four numbers $Z_j$, $|Z_j|=R$, is negative (positive), the principal pole-sets corresponding to $Z_1$ and $Z_2$ do (do not) separate on $m$ the principal pole-sets corresponding to $Z_3$ and $Z_4$.

*M. Marden* (Milwaukee, Wis.).

**v. Sz. Nagy, Gyula. Über die Lage der Nullstellen eines Abstandspolynoms und seiner Derivierten.** Bull. Amer. Math. Soc. **55**, 329–342 (1949).

By a distance-polynomial (Abstandspolynom) the author means the function $F(x, y, z)=C\prod_{k=1}^n d_k$, where

$$d_k=(x-x_k)^2+(y-y_k)^2+(z-z_k)^2$$

and $C>0$. By its derivative he means the polynomial of degree $2(n-1)$: $F'(x, y, z)=4F^{-1}H$, where $H=F_x^2+F_y^2+F_z^2$. Any zero of $F'(x, y, z)$, not also a zero of $F(x, y, z)$, satisfies simultaneously the equations $F_x/2F=\sum_{k=1}^n(x-x_k)/d_k=0$, $F_y/2F=\sum_{k=1}^n(y-y_k)/d_k=0$, $F_z/2F=\sum_{k=1}^n(z-z_k)/d_k=0$. Because of the similarity in form between these equations and those for the derivative of a polynomial in a complex variable, the distance-polynomials are found to possess properties analogous to those of the derivative of a polynomial in a complex variable. These include (1) a Lucas type theorem: all the zeros of $F'(x, y, z)$ lie in any convex region enclosing all the zeros of $F(x, y, z)$; (2) a Laguerre type theorem: the zeros of $F(x, y, z)$ are separated by any sphere which passes through a zero $(x, y, z)$ of $F(x, y, z)$ or $F'(x, y, z)$ and through the point $(\xi, \eta, \zeta)$ defined by the equations $\xi=x-2nH^{-1}F_x$, $\eta=y-2nH^{-1}F_y$, $\zeta=z-2nH^{-1}F_z$; (3) a Jensen type theorem: if the zeros of $F(x, y, z)$ are symmetric in a plane $E$, any zero of $F'(x, y, z)$ exterior to $E$ lies in at least one of the spheres having as diameters the line-segments joining the pairs of zeros of $F(x, y, z)$ sym-

metric in $E$. These theorems are proved in ways similar to those used in establishing the corresponding theorems on the derivative of a polynomial in a complex variable.

*M. Marden* (Milwaukee, Wis.).

**Pollaczek, Félix. Sur une généralisation des polynomes de Legendre.** C. R. Acad. Sci. Paris **228**, 1363–1365 (1949).

**Pollaczek, Félix. Systèmes de polynomes biorthogonaux à coefficients réels.** C. R. Acad. Sci. Paris **228**, 1553–1556 (1949).

A set of polynomials $\{P_n(x)\}$ is defined which are orthogonal on $-1 \leqq x = \cos\theta \leqq 1$ with weight function

$$\rho(x) = e^{\theta\varphi(\theta)}(1+e^{\pi\varphi(\theta)})^{-1},$$

where $\varphi(\theta)\sin\theta = a\cos\theta + b$ ($b$ real, $a \geqq |b|$). The definition is based on an elementary generating function $g(x; z)$ satisfying the differential equation

$$(1-2xz+z^2)dg/dz + \{z-(a+1)x-b\}g = 0.$$

The orthogonality relation is proved by applying an appropriate complex integration to the solution $u = u(x, y; z)$ of

$$(1-2xz+z^2)du/dz + \{z-(a+1)x-b\}u = -(1-yz)^{-1} \qquad (|y|<1,\ |z|<1),$$

which is regular in the complex $x$-plane cut along $(-1, 1)$. This solution is unique and can be represented in the form $\sum u_\nu(z)x^{-\nu-1}$, $\nu = 0, 1, 2, \cdots$. The residue $u_0(z)$ is then computed in two different ways, and this yields the orthogonality. The polynomials $P_n(x)$ satisfy the relation

$$(n+1)P_{n+1} - \{(2n+a+1)x+b\}P_n + nP_{n-1} = 0,$$

$P_0 = 1$, reducing to Legendre polynomials for $a = b = 0$. The second note generalizes the results of the first one by constructing two sets of polynomials which are "biorthogonal" on a system of disjoint real intervals with an appropriate weight function. The proofs are similar but somewhat more complicated than in the first case. *G. Szegö.*

**Sharma, A. On a generalisation of Legendre polynomials.** Bull. Calcutta Math. Soc. **40**, 195–206 (1948).

Let $\alpha$ be a primitive root of unity of order $k$. We form the operation $L_k f(z) = \sum_{r=0}^{k-1} \bar{\alpha}^r F(\alpha^r)$, $F^r(z) = f(z)$. A. Kharadze [cf. Mitt. Akad. Wiss. Georgischen SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 15–21 (1941); these Rev. **3**, 110] has considered the polynomial $Q_n(z)$ of degree $n$ for which $L_k Q_n(z)\varphi(z) = 0$ holds, $\varphi(z)$ being an arbitrary polynomial of degree $n-1$. All $Q_n(z)$ are identically zero except when $n$ or $n-1$ is divisible by $k$. The author discusses recurrence and related formulas for the polynomials $Q_{nk}(z)$ and $Q_{nk+1}(z)$ and the relation of these polynomials to Bessel functions. All this follows easily by representing these polynomials in terms of suitable Jacobi polynomials.

*G. Szegö* (Stanford University, Calif.).

**Tricomi, Francesco. Sul comportamento asintotico dell'$n$-esimo polinomio di Laguerre nell'intorno dell'ascissa $4n$.** Comment. Math. Helv. **22**, 150–167 (1949).

This paper deals with the asymptotic behavior of the Laguerre polynomials $L_n^{(\alpha)}(x)$ in the vicinity of $v = 4n+2(\alpha+1)$ as $n \to \infty$. The formula

$$(-1)^n e^{-\frac{1}{2}v} L_n^{(\alpha)}(x) = \gamma\{A(t) + B(t)v^{-\frac{1}{2}}\} + O(n^{-1})$$

is proved, where $x = v - (\frac{1}{2})^{\frac{1}{3}} v^{\frac{1}{3}} t$, $\gamma = av^{-\frac{1}{3}} + bv^{-1} + O(n^{-\frac{4}{3}})$. The function $A(t)$ is Airy's function (expressible in terms of $J_{\frac{1}{3}}$ and $J_{-\frac{1}{3}}$), the function $B(t)$ can be expressed in terms of $A(t)$ and $a$ and $b$ are fixed constants with $a > 0$. A consequence is the following asymptotic formula for the $r$th zero $\lambda_r$ of $L_n^{(\alpha)}(x)$ (ordered in decreasing order):

$$\lambda_r = v - \beta i_r v^{\frac{1}{3}} + \beta' i_r^2 v^{-\frac{1}{3}} + O(n^{-1}),$$

where $i_r$ is the $r$th positive zero of $A(t)$ and $\beta$ and $\beta'$ are absolute constants, both positive. The method used by the author is that of Liouville-Steklov-Fubini-Langer.

*G. Szegö* (Stanford University, Calif.).

**Kanter, Louis H. The zeros of the Jacobi polynomials and the corresponding Christoffel numbers.** Duke Math. J. **16**, 125–130 (1949).

Si $\{P_n(x)\}$ est le système des polynômes orthogonaux, dans l'intervalle $(a, b)$, par égard à le poids $w(x)$, et si $x_1, \cdots, x_n$ sont les racines de $P_n(x)$, on peut écrire la formule de quadrature de Gauss-Jacobi

$$\int_a^b \rho(x)w(x)dx = \sum_{r=1}^{n} \lambda_r \rho(x_r),$$

où $\rho(x)$ est un polynôme arbitraire de degré $2n-1$ et $\lambda_1, \cdots, \lambda_n$ sont les nombres de Christoffel. Si le poids $w(x)$ dépend aussi d'un paramètre $\tau$, on peut considérer le problème d'étudier la dépendance des nombres de Christoffel $\lambda_r$ du paramètre $\tau$. L'auteur étudie le problème dans le cas des polynômes de Jacobi (en particulier, des polynômes ultrasphériques) en donnant quatre théorèmes sur la décroissance des nombres $\lambda_r(\tau)$. Il donne aussi, dans le cas des polynômes ultrasphériques $[a = -1,\ b = 1,\ w = (1-x^2)^{\lambda-\frac{1}{2}},\ \lambda > -\frac{1}{2}]$, des bornes pour le nombre $\lambda_1(\lambda)$ qui correspond à la plus grande racine $x_1$. *A. Ghizzetti* (Pisa).

**Geronimus, Ya. L. On some quadrature formulas.** Doklady Akad. Nauk SSSR (N.S.) **65**, 437–440 (1949). (Russian)

The paper contains generalisations of S. N. Bernstein's theorem on the quadrature formula of Čebyšev [C. R. (Doklady) Acad. Sci. URSS (N.S.) **14**, 323–326 (1937)]: the maximum degree $M_n$ for which the formula

$$\int_a^b f(x)dx = n^{-1}\sum_{i=1}^{n} f(x_i), \qquad a \leqq x_i \leqq b,$$

is correct for all polynomials satisfies $M_n < 4n^{\frac{1}{2}}$. The author considers

$$\int_a^b f(x)d\sigma(x) = \sum_{i=1}^{n} \lambda_i^{(n)} f(y_i^{(n)}); \qquad \lambda_i^{(n)} > 0,\ a \leqq y_i^{(n)} \leqq b.$$

Constructing the staircase-function $\psi_n$ having a jump $1/n$ at the zeros of $P_n(x)$, the orthogonal polynomials for $d\sigma(x)$, the limit function $\psi(x)$ of the zeros is constructed: the function $\sigma(x)$ is said to belong to the class $A$ if (1) the points of discontinuity are on a closed set of positive measure $E_0 \subset [a, b]$; (2) the limit function $\psi(x)$ satisfies

$$\int_{E_0} \log(1/|z-x|)d\mu(x) = \text{constant},\ z \subset E_0; \qquad \int_{E_0} d\mu(x) = 1.$$

His results are as follows. (1) If $e = [a, a+\epsilon] \subset E_0$ then $\mu(x) = O((x-a)^{\frac{1}{2}})$, $x \subset e$. (2) if $\sigma(x) \subset A$ is continuous and $\sigma'(x) > 0$ almost everywhere on $e$, whereas

$$\lim_{x \to a} \sigma(x)/(x-a)^{\frac{1}{2}} = 0,$$

then

$$\lim_{n \to \infty} \{M_n \lambda_1^{(n)}\} = 0.$$

In a special case the function $\mu(x)$ is given explicitly.

*E. M. Bruins* (Amsterdam).

## *Special Functions*

**Hillman, Abraham. On the reality of zeros of Bessel functions.** Bull. Amer. Math. Soc. **55**, 198–200 (1949).

The functions $J_\nu(z)$ and $Y_\nu(z)$ ($\nu$ real) are in general many-valued. If $z$ is replaced by the positive real variable $x$ and the principal value of $(\frac{1}{2}x)^\nu$ is used, real-valued functions $J_\nu(x)$ and $Y_\nu(x)$ are obtained. All branches of any Bessel function $B(z)$ can be obtained by analytic continuation of a function $B(x)=(a+ib)J_\nu(x)+(h+ik)Y_\nu(x)$, where $a$, $b$, $h$, and $k$ are real numbers. In particular, let $B(x, m)$ stand for the result of continuing $B(x)$ through an angle of $m\pi$ along a circle with center at the origin ($m$ an integer). It can be shown that

$$B(x,m)=[(aC+bS-2kS\cot\nu\pi)J_\nu(x)+(hC+kS)Y_\nu(x)] +i[(bC+aS+2kS\cot\nu\pi)J_\nu(x)+(kC-hS)Y_\nu(x)],$$

where $C=\cos m\nu\pi$ and $S=\sin m\nu\pi$. A necessary and sufficient condition for $B(x, m)$ to have zeros is that

$$\Delta=[ak-bh]\cos 2m\nu\pi-[ah+bk+(h^2+k^2)\cot\nu\pi]\sin 2m\nu\pi$$

is equal to zero. It is noted that every Bessel function has, on every branch and in both the left and the right half-planes, an infinite number of zeros which lie on or approach a horizontal straight line (the line depends on the branch and the half-plane). Many special cases are examined.

*S. C. van Veen* (Delft).

**Fasenmyer, Mary Celine. A note on pure recurrence relations.** Amer. Math. Monthly **56**, 14–17 (1949).

The method is illustrated by the derivation of a pure recurrence relation for Bateman's

$$Z_n(s)={}_2F_2\left[\begin{matrix}-n, n+1\\ 1,\ 1\end{matrix};s\right] =\frac{1}{\Gamma(-n)\Gamma(n+1)}\sum_{r=0}^{\infty}\frac{\Gamma(r-n)\Gamma(r+n+1)}{(r!)^2}s^r=\sum_{r=0}^{\infty}\beta_r.$$

By listing a sequence of the $Z_k(s)$ and $sZ_k(s)$ ($k=n, n-1, n-2, n-3$) the coefficients of $\beta_r$ are written with a least common denominator. The numerators are polynomials of degree at most four in $r$. A linear combination of the six series $Z_n(s)$, $Z_{n-1}(s)$, $Z_{n-2}(s)$, $Z_{n-3}(s)$, $sZ_{n-1}(s)$ and $sZ_{n-2}(s)$ will have, as numerator of its coefficients of $\beta_r$, a polynomial of degree four in $r$. Such a linear combination will leave five undetermined constants with which to make the five coefficients in the fourth degree polynomial vanish. This suggests that, for $n\geqq 3$, there exists a linear recurrence relation of the form

$$Z_n(s)+(A+Bs)Z_{n-1}(s)+(C+Ds)Z_{n-2}(s)+EZ_{n-3}(s)=0,$$

in which $A$, $B$, $C$, $D$, and $E$ are rational functions in $n$ and are independent of $s$. For $n\geqq 3$ the four term recurrence relation

$$n^2(2n-3)Z_n-(2n-1)[3n^2-6n+2-2(2n-3)s]Z_{n-1} -(2n-3)[3n^2-6n+2+2(2n-1)s]Z_{n-2} -(2n-1)(2n-2)^2Z_{n-3}=0$$

is readily determined. In the same way for a special case of a set of generalised hypergeometric polynomials

$$f_n(a;-;x)={}_3F_2\left[\begin{matrix}-n, n+1, a\\ \frac{1}{2},\ 1\end{matrix};x\right] =\frac{\Gamma(\frac{1}{2})}{\Gamma(-n)\Gamma(n+1)\Gamma(a)}\sum_{r=0}^{\infty}\frac{\Gamma(r-n)\Gamma(r+n+1)\Gamma(r+a)}{\Gamma(r+\frac{1}{2})(r!)^2}x^r$$

($a\neq\pm\frac{1}{2}$), the relation is

$$nf_n-[(3n-2)-4(n-1+a)x]f_{n-1} +[(3n-4)-4(n-1-a)x]f_{n-2}-(n-2)f_{n-3}=0$$

for $n\geqq 3$.

*S. C. van Veen* (Delft).

**Karlin, Meyer. Note on the expansion of confluent hypergeometric functions in terms of Bessel functions of integral order.** J. Math. Physics **28**, 43–44 (1949).

The well-known fact that any absolutely convergent power series can be expressed as a series of Bessel functions of integral order gives

$$F(\alpha,\gamma;x)=\sum_{k=0}^{\infty}\frac{\Gamma(\gamma)\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(\gamma+k)}\frac{x^k}{k!}=\sum_{k=0}^{\infty}c_kI_k(x).$$

The boundary conditions yield $c_0=1$, $c_1=2\alpha/\gamma$. By means of the recurrence relations of $I_k(x)$, the author finds the following recurrence relations for $c_k$: $\frac{1}{2}(c_{k+1}+c_{k-1})=\alpha d_k/\gamma$ for $k\geqq 2$; $\frac{1}{2}c_2+1=2\alpha(\alpha+1)/\gamma(\gamma+1)$; where $d_k$ is obtained from $c_k$ by replacing $\alpha$ and $\gamma$ by $\alpha+1$ and $\gamma+1$, respectively, from which the expansion is obtained. As an illustration, for $\alpha=4\frac{1}{2}$, $\gamma=\frac{1}{2}$, the author obtains

$$F(4\tfrac{1}{2},\tfrac{1}{2};x)=I_0(x)+82I_1(x)+\tfrac{7046}{3}I_2(x)+42066I_3(x) +\tfrac{1178470}{11}I_4(x)+\tfrac{1702874}{3}I_5(x)+\tfrac{41566910842}{77}I_6(x) +\tfrac{6678482795}{143}I_7(x)+\cdots.$$

Eight terms in this expansion yield $F(20.5, \frac{1}{2}; 2)=30.97745$ as compared with the exact value 30.97819. To obtain a comparable accuracy from the power series, hundreds of terms would be needed.

*S. C. van Veen* (Delft).

**Straubel, Rudolf. Unbestimmte Integrale mit Produkten von Zylinderfunktionen. I.** Ing.-Arch. **12**, 325–336 (1941).

The author lists a number of relations between indefinite integrals containing products of Bessel functions. His formulas usually contain two, possibly different, Bessel functions $C_p$ and $D_q$ of orders $p$, $q$ and variable $z$, and also one or two arbitrary differentiable functions. One of the simpler relations is

$$2\int f_1C_p'D_q\,dz+\int\{(p^2-q^2)f_0+f_2\}C_pD_qz^{-1}dz =zf_0(C_p'D_q-C_pD_q')+f_1C_pD_q,$$

where $f_0$ is an arbitrary twice differentiable function of $z$, and $f_n=zf_{n-1}$. The proofs flow from the recurrence and differentiation formulas for Bessel functions.

*A. Erdélyi* (Pasadena, Calif.).

**Lord, R. D. Some integrals involving Hermite polynomials.** J. London Math. Soc. **24**, 101–112 (1949).

This paper belongs to a series which was started by Titchmarsh [same J. **23**, 15–16 (1948); these Rev. **10**, 115] and continued by Busbridge [same J. **23**, 135–141 (1948); these Rev. **10**, 296] and Bailey [same J. **23**, 291–297 (1948); these Rev. **10**, 454]. The author evaluates the integral

$$I_{mn\cdots}^{ab\cdots}=\int_{-\infty}^{\infty}e^{-x^2}H_m(ax)H_n(bx)\cdots dx$$

in terms of Lauricella's hypergeometric series $F_B$. He is primarily interested in the cases when the result simplifies, and also in simple methods of proof. "In this paper I give a few new results, but my chief aim is to give simple proofs by the method of repeated integration by parts . . . and by the method of integrating the generating function."

*A. Erdélyi* (Pasadena, Calif.).

**Cansado Maceda, E. Characteristic functions of the Pearson distributions. II.** Revista Mat. Hisp.-Amer. (4) **8**, 203–225 (1948). (Spanish)

[For the first part cf. the same Revista (4) **7**, 117–127 (1947); these Rev. **9**, 283.] In this part the author derives the characteristic functions of the remaining types of Pearson's distributions. For types IV, VI, XI, the characteristic function is expressed in terms of Whittaker's confluent hypergeometric function $W_{k,m}(z)$; for types V, VII, in terms of Bessel functions. Several particular cases are also considered, with the simplified formulas appertaining to them. It is also shown how to obtain the characteristic functions directly from the differential equation satisfied by the Pearson distributions. The characteristic function of Student's $t$ is expressed in terms of Bessel functions, and the (elementary) characteristic function of Cauchy's distribution is also evaluated. *A. Erdélyi* (Pasadena, Calif.).

**Hesselbach, B. Ein Vertauschungssatz für elliptische Integrale erster Gattung.** Math. Ann. **121**, 33–40 (1949).

The author proves that the congruence

$$\int_p^q \{(t-u)(t-v)(t-w)(t-z)\}^{-\frac{1}{2}}dt \equiv \int_r^s \{(t-u)(t-v)(t-w)(t-z)\}^{-\frac{1}{2}}dt \pmod{K}$$

entails

$$\int_\alpha^\beta \{(t-p)(t-q)(t-r)(t-s)\}^{-\frac{1}{2}}dt \equiv \pm\int_\omega^\sigma \{(t-p)(t-q)(t-r)(t-s)\}^{-\frac{1}{2}}dt \pmod{L},$$

where $K$ and $L$, respectively, are the period systems of the two elliptic integrals. The proof is based on geometric properties of conics. *Z. Nehari* (St. Louis, Mo.).

**Meixner, J. Lamé's wave functions of the ellipsoid of revolution.** Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1224**, iii+102 pp. (1949).

[Translated from Zentrale für Wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters (ZWB), Forschungsber. **1952** (1944).] The main purpose of this report is to correlate the known results on spheroidal wave functions, and make these functions accessible for practical use. In the course of the work the author finds many occasions to simplify proofs, to fill in gaps, to generalise and to extend results, and in many ways to contribute materially to the theory. The standard differential equation is

$$(1)\qquad \frac{d}{d\xi}\left\{(1-\xi^2)\frac{dy}{d\xi}\right\}+\{\lambda-\gamma^2\xi^2-\mu^2/(1-\xi^2)\}y=0,$$

and it originates when the wave equation is separated in spheroidal coordinates. The relation of this to the Legendre equation is obvious, and connexions with the Mathieu, Bessel, Laguerre, and Hermite equations can be established.

At first $\mu$ and $\lambda$ are unrestricted and $\xi$ is a complex variable. Solutions of (1) can be expanded as infinite series of $P^{\mu}_{\nu+r}(\xi)$ or $Q^{\mu}_{\nu+r}(\xi)$, where $\mu$, $\nu$ are fixed and $r$ runs through all integers. These solutions are known as the $X$-functions of the first and second kind, respectively. The recurrence relations satisfied by the coefficients and the convergence of the series are investigated. Known relations between Legendre functions induce corresponding relations between $X$-functions. The parameter $\lambda$ in (1) is a one-valued function of $\gamma$, $\mu$, $\nu$.

The expansion of the solutions of (1) in terms of Bessel functions of variable $\xi$ introduces $Z$-functions of which there are four kinds. These are similarly investigated, as are the relations connecting $X$-functions and $Z$-functions. Alternative expansions in terms of Bessel functions of variable $\gamma(\xi^2-1)^{\frac{1}{2}}$ are briefly mentioned. Considerable attention is paid to the computation of $\lambda$ and of the coefficients of the various expansions ("coefficients" shortly) both by continued fraction methods and by series of powers of $\gamma^2$.

In the second part of the monograph $\mu$ and $\nu$ are integers $m$ and $n$, respectively, and $n\geqq m\geqq 0$. Among the topics discussed are the special form of the expansions in this case, orthogonal property, zeros, integral equations. The asymptotic behaviour of the characteristic values of $\lambda$ and of the characteristic functions is given for large $n$ (or $\lambda$), and also for large $\gamma$ (real or imaginary). The discussion of appropriate wave functions for the prolate and oblate spheroid leads to the expansion of spherical and plane waves in terms of spheroidal waves, and to an indication of the use of these expansions in diffraction theory.

There are numerical tables giving expansions of the $\lambda^m_n(\gamma)$ in powers of $\gamma^2$ up to $\gamma^{10}$ for $0\leqq m\leqq n\leqq 9$, expansions of the coefficients up to $\gamma^6$, some numerical values of characteristic numbers and of coefficients taken from Bouwkamp's thesis [Groningen, 1941; these Rev. **8**, 179], and graphs of some characteristic values plotted against $\gamma^2$.

*A. Erdélyi* (Pasadena, Calif.).

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### *Harmonic Functions, Potential Theory*

**Heilbronn, H. A. On discrete harmonic functions.** Proc. Cambridge Philos. Soc. **45**, 194–206 (1949).

Il s'agit des fonctions "discrètes-harmoniques," ou "préharmoniques," notion aujourd'hui classique, ayant fait l'objet de nombreux exposés. L'auteur [qui ne donne aucune indication bibliographique] s'intéresse notamment aux polynomes préharmoniques, à l'approximation des fonctions préharmoniques par de tels polynomes (une fonction préharmonique dans un domaine borné convexe y est égale à un polynome préharmonique), aux fonctions préharmoniques ayant un point singulier isolé, au problème de Dirichlet pour fonctions bornées et domaines non bornés (l'unicité n'est plus assurée si le nombre des variables est $>2$), à l'étude d'un noyau analogue à celui de Poisson, et à quelques inégalités qui s'en déduisent. *H. Cartan.*

**Tautz, Georg. Zur Theorie der ersten Randwertaufgabe.** Arch. Math. **1**, 312–316 (1949).

L'auteur reprend la méthode de Perron qui résoud le problème de Dirichlet, pour en dégager, dans un aperçu rapide, des principes axiomatiques permettant un développement analogue, non seulement pour des équations du type elliptique, mais en espace métrique localement compact, l'intégrale de Poisson étant remplacée par un opérateur satisfaisant à certaines conditions. On introduit une solution fondamentale, une fonction barrière et dans l'espace euclidien on généralise le critère de régularité de Wiener.

*M. Brelot* (Grenoble).

**Rauch, H. E. Generalizations of some theorems of R. Nevanlinna.** Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 51, 7 pp. (1948).

Démonstration des deux théorèmes suivants (formulés pour l'espace euclidien $E_n$, mais susceptibles d'être étendus aux variétés riemanniennes). Soit $G$ un domaine borné de $E_n$ limité par des hypersurfaces fermées régulières, dont l'une est appelée $\gamma_0$, l'ensemble des autres étant noté $\gamma$. Pour toute $f$ harmonique dans $G$ et nulle sur $\gamma_0$, soit $\|f\|^2$ l'intégrale de Dirichlet $\int_G |\text{grad } f|^2 dv = \int_\gamma f(\partial f/\partial n) d\sigma$. Théorème 1. Dans l'ensemble $H$ des $f$ harmoniques dans $G$, nulles sur $\gamma_0$ et telles que $\int_{\gamma_0} |\partial f/\partial n| d\sigma = 1$, le minimum de $\|f\|$ est atteint pour la fonction $\pm\omega$ qui est constante sur $\gamma$. Théorème 2 (conséquence du théorème 1). Toute fonction harmonique dans la région de $E_n$ extérieure à une hypersurface fermée $\gamma_0$ a une intégrale de Dirichlet finie (si $E_n$ est remplacé par une variété riemannienne, on suppose qu'elle a une "frontière nulle"). La démonstration du théorème 1 se partage en deux parties. (1) Soit $H^+$ l'ensemble des $f$ harmoniques $\geqq 0$ dans $G$, nulles sur $\gamma_0$ et telles que $\int_\gamma (\partial f/\partial n) d\sigma = -1$; le minimum de $\|f\|$ est atteint pour la fonction $\omega$ constante sur $\gamma$ (car $\omega$ est orthogonale à $f-\omega$ pour toute $f$ de $H^+$). (2) Soit $g$ harmonique dans $G$ et nulle sur $\gamma_0$; la fonction harmonique $f$, nulle sur $\gamma_0$, égale à $|g|$ sur $\gamma$, satisfait à $-\partial f/\partial n \geqq |\partial g/\partial n|$ sur $\gamma_0$, $\|f\| \leqq \|g\|$.

*H. Cartan* (Paris).

**Kawata, Tatsuo. The harmonic functions in a half-plane and Fourier transforms.** Kōdai Math. Sem. Rep., no. 1, 9–15 (1949).

L'auteur étend aux fonctions harmoniques un résultat démontré antérieurement pour les fonctions analytiques [Hille et Tamarkin, Fund. Math. **25**, 329–352 (1935); l'auteur, Jap. J. Math. **13**, 421–430, 483–491 (1937)]: si $f(x, y)$ est harmonique dans le demi-plan $y>0$ et satisfait à

$$\int_{-\infty}^{\infty} |f(x+iy)|^p dx \leqq C, \qquad p>0,$$

alors $f(x, y)$ converge vers une fonction $f(x)$ lorsque $y \to 0$, pour presque tout $x$; et on a:

$$f(x, y) = \pi^{-1} \int_{-\infty}^{\infty} \frac{y}{(\xi-x)^2+y^2} f(\xi) d\xi.$$

On en déduit, par analogie avec ce qui a été fait par Verblunsky [Proc. London Math. Soc. (2) **38**, 1–48 (1934)] le résultat suivant. Soit $g(u)$ telle que $\int_{-\infty}^{\infty} e^{-\varepsilon|u|} |g(u)| du < \infty$ pour tout $y>0$ et posons $f(x, y) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\varepsilon|u|} e^{ixu} g(u) du$. Si $\int_{-\infty}^{\infty} |f(x, y)|^p dx \leqq C$ $(p \geqq 1)$ avec pour $p=1$ la condition supplémentaire $\int_e |f(x, y)| dx \leqq \epsilon$ pour tout $e$ tel que $m(e) \leqq \delta(\epsilon)$, alors il existe une fonction $f(x) \varepsilon L_p$ telle que $f(x, y)$ converge vers $f(x)$ pour presque tout $x$ et que l'on ait

$$f(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} g(u) e^{-ixu} du, \quad g(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x) e^{ixu} dx$$

presque partout. Si $p>1$, $f(x)$ est la transformée de Fourier de $g(u)$ dans $L_p$. Ce résultat peut être généralisé, selon une idée due à Offord [Trans. Amer. Math. Soc. **38**, 250–266 (1935)] aux fonctions $g(u)$ telles que

$$f_\omega(x, y) = (2\pi)^{-\frac{1}{2}} \int_{-\omega}^{\omega} (1-|u|/\omega) e^{-\varepsilon|u|} g(u) e^{-ixu} du$$

satisfasse aux conditions imposées précédemment à $f(x, y)$.

*P. Lelong* (Lille).

**Manacorda, T. Sulle discontinuità delle derivate del potenziale di semplice strato. I.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 14–19 (1948).

L'auteur s'occupe de la détermination des discontinuités des dérivées troisièmes du potentiel de simple couche sur la surface qui engendre le champ. Il profite de l'algorithme des homographies vectorielles et il calcule une certaine homographie $\varphi$ au moyen de laquelle on peut exprimer les susdites discontinuités. Les expressions explicites sont données dans la deuxième partie du travail [voir ci-après].

*A. Ghizzetti* (Pisa).

**Manacorda, T. Sulle discontinuità delle derivate del potenziale di semplice strato. II.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 143–147 (1948).

En poursuivant la recherche de la note I [voir l'analyse ci-dessus], l'auteur donne les expressions des discontinuités des dix dérivées troisièmes d'un potentiel de simple couche sur la surface qui engendre le champ. Il donne aussi des applications à des cas particuliers. *A. Ghizzetti* (Pisa).

**Gans, Richard. Lecher-System in einer Schutzhülle.** Z. Naturforschung **3a**, 519–521 (1948).

By the use of conformal transformations the author obtains a formula for the mutual capacitance between the two wires of a Lecher system when the system is surrounded by a cylindrical screen. Mathematically the problem is that of finding a solution of Laplace's equation, $\Delta\phi = 0$, subject to the boundary conditions that $\phi = \pm U$ (a known potential) on the surfaces of the two wires of radius $r$, and $\phi = 0$ on the surface of the screen of radius $R$. *M. C. Gray.*

**Miranda, Carlo. Formule di maggiorazione e teorema di esistenza per le funzioni biarmoniche di due variabili.** Giorn. Mat. Battaglini (4) **2(78)**, 97–118 (1948).

Let $T$ be a bounded region in the plane whose $n+1$ bounding curves have continuous curvature. Let $u(x, y)$ be a function continuous with its first derivatives in $T$, satisfying the biharmonic equation $\Delta^2 u = 0$ interior to $T$. Let $f$ and $g$ be continuous functions of arc length on the boundary $FT$ of $T$ such that on $FT$ one has $u = f$ and $\partial u/\partial \nu = g$, where $\nu$ is the interior normal to $T$, and let $f'$ be continuous. The author obtains bounds for $|u|$ and $(u_x^2+u_y^2)^{\frac{1}{2}}$ in terms of the maximum values of $|f|$, $|f'|$, $|g|$ on $FT$; the most important is of the form

$$(u_x^2+u_y^2)^{\frac{1}{2}} \leqq K_1 \{\max |g| + \max |f'|\} + K_2 \max |f|,$$

where $K_1$ and $K_2$ depend only on $T$. In the case $n=0$, the value of $K_2$ can be taken to be zero.

Using these a priori bounds, a short proof of the existence of a unique solution of the biharmonic equation for given $f$ and $g$ is carried out by considering the linear space of functions pairs $(f, g)$ on $FT$ with $f$, $f'$, $g$ continuous and norm given by $\max |f| + \max |f'| + \max |g|$. It follows that those pairs for which the biharmonic equation has a solution form a closed linear subspace. Unless this subspace is the whole space, there exists a linear functional which vanishes on the subspace but not identically. This is shown to be impossible by expressing the functional as a Stieltjes integral over $FT$ and considering known solutions of the biharmonic equation. A final section is devoted to the

approximation to arbitrary biharmonic functions by series of biharmonic functions of special type; they are polynomials in the case $n=0$. *J. W. Green.*

**De Cicco, John. Functions of several complex variables and multiharmonic functions.** Amer. Math. Monthly **56**, 315–325 (1949).

Propriétés des parties réelles (fonctions multiharmoniques) des fonctions analytiques $F$ de $n$ variables complexes. Conditions de Cauchy, système différentiel de Poincaré. Détermination de $F$ à partir de sa partie réelle: $F$ est algébrique, ou entière en même temps que celle-ci. Une condition nécessaire et suffisante pour que $f(x_i, y_i)$ soit multiharmonique est que sa trace sur toute variété analytique $W^1$ à une dimension complexe soit une fonction harmonique. Ce critère a déjà été donné en application de résultats généraux [cf. P. Lelong, Ann. Sci. École Norm. Sup. (3) **62**, 301–338 (1945); ces Rev. **8**, 271].

*P. Lelong* (Lille).

**Rafal'son, È. H. Concerning the solution of the biharmonic equation.** Doklady Akad. Nauk SSSR (N.S.) **64**, 799–802 (1949). (Russian)

L'auteur considère un domaine plan simplement connexe $D$, de contour $L$ formé d'arcs à tangente continue. Il cherche une fonction $u$ à dérivées secondes continues dans $D$, à dérivées premières continues dans $D+L$, ayant un $\Delta u$ de carré sommable et satisfaisant à $\Delta\Delta u=f$ höldérienne, avec $u$ et $du/dn$ nuls au contour. Introduisons dans l'espace $L_2$ des fonctions de carré sommable dans $D$ le sous-espace $G_2$ des fonctions harmoniques. Si $p\in L_2$, il se décompose en $p_\sigma+p_{\bar\sigma}$ ($p_\sigma\in G_2$; $p_{\bar\sigma}$ dans la variété $\bar G_2$ orthogonale à $G_2$). Soit alors $q(x, y)$ quelconque de dérivées premières continues dans $D$, satisfaisant à $\Delta q=f$. Il est montré sommairement que si $u$ est la solution cherchée, $\Delta u=q_{\bar\sigma}$ et

$$u(x, y)=(2\pi)^{-1}\int\int q_{\bar\sigma}(\xi, \eta)\log r\, d\xi d\eta,$$
$$r=\{(x-\xi)^2+\cdots\}^{\frac{1}{2}}.$$

De là est tirée inversement la solution du problème initial, à cela près que la condition $du/dn=0$ à la frontière est remplacée par une autre voisine mais globale et moins précise. On complète par une étude variationnelle. Il est énoncé que $\int\int\{(\Delta v)^2-2fv\}d\sigma$ pour $v$ et $dv/dn$ nuls à la frontière, atteint pour $v=u$ solution du problème initial, son minimum égal à $-\int\int q_{\bar\sigma}^2 d\sigma$ ($q$ quelconque considérée plus haut). Application à la représentation de la solution $u$. Indication d'extension possible à l'espace.

*M. Brelot* (Grenoble).

**Pescarini, Angelo. Su una certa classe di sistemi di equazioni alle derivate parziali, lineari del 1° ordine.** Rend. Sem. Fac. Sci. Univ. Cagliari **17** (1947), 261–268 (1948).

L'auteur considère des systèmes de $2k$ équations aux dérivées partielles du premier ordre, homogènes, linéaires, à coefficients constants, avec $2k$ fonctions inconnues. Un petit calcul donne une condition nécessaire et suffisante pour que les solutions soient toutes polyharmoniques d'ordre $k$ et on montre que certaines conditions intégrales (généralisant celle de Morera qui traduit l'holomorphie) entrainent que $2k$ fonctions soient solutions du système et polyharmoniques d'ordre $k$.

*M. Brelot* (Grenoble).

## *Differential Equations*

**Kasner, Edward, and Mittleman, Don. Second order differential equations of rank 2 in physics.** Proc. Nat. Acad. Sci. U. S. A. **35**, 338–342 (1949).

If we introduce the homogeneous variables

$$z_1:z_2:z_3:z_4:z_5=1:y':y'^2:y'^3:y'',$$

ordinary differential equations of the second order which are algebraic in $y'$ and $y''$ can be classified according to the degrees of the corresponding equations in the $z$'s. The degree of the equation in the $z$'s will be called the rank of the differential equation. This note is devoted to a discussion of equations of rank 2, i.e., equations of the form

$$A_3y''^2+(B_0+B_1y'+B_2y'^2+B_3y'^3)y''$$
$$+(C_0+C_1y'+C_2y'^2+C_3y'^3+C_4y'^4+C_5y'^5+C_6y'^6)=0,$$

where the coefficients are arbitrary functions of the independent variable $x$ and the dependent variable $y$. The principal result is the following. In general the locus of the centers of curvature, at a given point, of the $2\infty^1$ integral curves passing through the point is a special sextic, characterized by the fact that the given point is a quadruple point, the tangent lines are the two minimal lines, each counted twice, and these tangents intersect the sextic only at the given point. The corresponding properties for some degenerate cases are discussed; and various problems which give rise to differential equations of rank 2 are indicated.

*L. A. MacColl* (New York, N. Y.).

**Flügge-Lotz, I., und Klotter, K. Über Bewegungen eines Schwingers unter dem Einfluss von Schwarz-Weiss-Regelungen. I. Bewegungen eines Schwingers von einem Freiheitsgrad; Regelung mit Stellungszuordnung ohne Schaltverschiebungen.** Z. Angew. Math. Mech. **28**, 317–337 (1948). (German. Russian summary)

This paper investigates the motion of a system which is controlled by a regulator of the "on-off" type. The differential equation of the system is taken as $A\ddot x+B\dot x+Cx=Ny$, with the regulator providing the additional relation

$$y=\pm y_0 \operatorname{sgn} F,$$

where $F=x+a\dot x$. After the elimination of $y$, the differential equation may be reduced to the normal form

$$\ddot x+2D\omega\dot x+\omega^2x=b \operatorname{sgn} F.$$

Between successive zeros of $F$, the real solutions of this equation and the corresponding $F$ may be written in the forms

$$x(t)=Ke^{-\delta t}\cos(\nu t+\epsilon)+(b/\omega^2)\operatorname{sgn} F,$$
$$F(t)=K'e^{-\delta t}\cos(\nu t+\epsilon')+(b/\omega^2)\operatorname{sgn} F.$$

The nature of the solution $x(t)$ is determined by graphs drawn in the phase space $(x, \dot x)$ where the zeros of $F$ appear as points on a line through the origin. The authors refer to H. Bilharz [same Z. **22**, 206–215 (1942); these Rev. **5**, 6] for a different treatment of the problem limited to the case $b<0$. *M. Marden* (Milwaukee, Wis.).

**Päsler, Max. Resonanzkurven von erzwungenen Schwingungen, erregt durch Störungen mit frequenzabhängiger Amplitude.** Ann. Physik (6) **4**, 1–13 (1948).

Assuming that the amplitude of the forcing term is proportional to the $n$th power of the frequency, the author studies the response of a simple damped oscillating system. He plots families of curves for $n=0$, 1, 2, 3, and 4 and discusses the variation of the response. *P. Franklin.*

**Constantinescu, G. G. Sur le critère de M. Pompeiu pour l'intégrabilité par quadratures de l'équation de Riccati.** Acad. Roum. Bull. Sect. Sci. **24**, 397–400 (1943).

The author shows that the Riccati equation

$$u' = a(x)u^2 + 2b(x)u + c(x)$$

is integrable by quadratures if there is a function $\phi(x)$ such that $\phi^{-1}\phi'' = A^{-1}A'' - a^{-1}[ab' - a'b + a(ac - b^2)]$, in which $A = a^{-1}$. If $\phi = A$, we get a criterion due to Pompeiu [C. R. Acad. Sci. Paris **161**, 235–237 (1915)].

*J. E. Wilkins, Jr.* (Buffalo, N. Y.).

**Gonzalez Baz, Enriqueta. The existence theorem for a system of an arbitrary number of differential equations of the first order.** Comisión Impulsora y Coordinadora de la Investigación Científica. (Mexico). Anuario **1946**, 49–55 (1947). (Spanish)

This proof of the existence of a unique solution of a system of ordinary differential equations follows the classical lines of the method of successive approximations.

*F. Bohnenblust* (Pasadena, Calif.).

**Peyovitch, T. L'existence de solutions asymptotiques de certaines équations différentielles.** Acad. Serbe Sci. Publ. Inst. Math. **1**, 88–92 (1947).

The paper deals with a differential system

$$dx/dt = -l + a_{11}(t)e^x + a_{12}(t)e^y, \quad dy/dt = m + a_{21}(t)e^x + a_{22}(t)e^y,$$

in which $l, m$ are constants and the $a_{ij}(t)$ are continuous functions which are bounded for $t \geqq t_0 > 0$. Under an elaborate set of conditions it is shown by successive approximation that there exists a one-parameter family of solutions for which $\lim_{t\to\infty} dx/dt = -l$, $\lim_{t\to\infty} dy/dt = m$.

*R. E. Langer* (Madison, Wis.).

**Lewis, D. C. Metric properties of differential equations.** Amer. J. Math. **71**, 294–312 (1949).

Let $x(t) = (x_i(t))$, $y(t) = (y_i(t))$ be two solutions of the system $\dot{x}_i = X^{(i)}(x, t)$. Denote by $D(t)$ the distance between $x(t)$ and $y(t)$. Estimates of the form

$$D(t_0)\exp\int_{t_0}^{t}\alpha(u)du \leqq D(t) \leqq D(t_0)\exp\int_{t_0}^{t}\beta(u)du$$

are obtained. The distance $D(t)$ can be measured as the geodesic distance of an underlying Finsler space. In the special case of the Euclidean metric, the functions $\alpha(t)$ and $\beta(t)$ are determined as the minimum and the maximum, respectively, of the quadratic form $Q(\lambda) = \sum X_i^j \lambda^i \lambda^j$ over $\sum \lambda^i \lambda^i = 1$, along the straight line from $x(t)$ to $y(t)$. In some cases this quadratic form can be negative definite and the above estimate leads then to an asymptotic approach of the solutions. The proofs assume that the $X^{(i)}$ are of class $C^1$ and that the underlying Finsler metric space is described by a function of class $C^2$. In addition to the above inequality, estimates on the length of curves, rather than geodesics, joining $x(t)$ and $y(t)$ are considered. Examples are given showing that the results are best of their type.

*F. Bohnenblust* (Pasadena, Calif.).

**Gusarova, R. S. On bounded solutions of a linear differential equation with periodic coefficients.** Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 241–246 (1949). (Russian)

Applying a geometric method originally used by Žukovskiĭ [Mat. Sbornik (1) **16**, 582–591 (1891)] to prove the initial theorem of Liapounoff concerning the boundedness of solutions of the equation $y'' + p(t)y = 0$, $p(t)$ periodic of period $\pi$, the author gives a proof of a theorem of Wallach [Proc. Nat. Acad. Sci. U. S. A. **34**, 203–204 (1948); these Rev. **9**, 509]. Borg's theorem, of which Wallach's is a sharpening, was proved by an application of the calculus of variations, while Wallach's and Gusarova's methods are more elementary. The author then derives some new results involving conditions on $\int_0^\pi p(t)dt$. Used in the proof of these results are theorems due to Hamel [Math. Ann. **73**, 371–412 (1913)] and Haupt [Math. Ann. **79**, 278–285 (1918)], and a variational technique involving the integral $\int_0^\pi \{y'^2 - p(t)y^2\}dt$.

*R. Bellman* (Stanford University, Calif.).

**Kalinin, S. V. On the stability of periodic motions in the case when one of the roots is zero.** Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 247–252 (1949). (Russian)

Consider the system of the classical Liapounoff type

$$(1) \qquad dx/dt = X, \quad dx_i/dt = \sum p_{ik}x_k + X_i$$

$(i = 1, 2, \cdots, n)$, where $X$, $X_i$ are holomorphic in the coordinates in a certain spherical region and for $t \geqq t_0$, begin with terms of degree at least 2 and have their coefficients periodic in $t$ with period $\omega$. Moreover, $p_{ik} = c_{ik} + \epsilon f_{ik}(t)$, where the $c_{ik}$ are constants, $\epsilon$ is a small parameter and the $f_{ik}$ have the same periodicity property as above. Finally it is assumed that the characteristic roots of $\|c_{ik}\|$ all have negative real parts. In a previous note [same journal **12**, 671–672 (1948); these Rev. **10**, 377] the author discussed the stability of (1) when the lowest power of $x$ at the right was two. In the present note the same problem is discussed when the lowest power of $x$ is an arbitrary positive integer, at least 2. The method used throughout is the so-called second method of Liapounoff [Ann. Fac. Sci. Univ. Toulouse (2) **9**, 203–474 (1907), reprinted as Ann. of Math. Studies, no. 17, Princeton University Press, 1947; these Rev. **9**, 34; complementary reference: Ch. Nougmanova, C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 202–204 (1944); these Rev. **6**, 154].

*S. Lefschetz* (Princeton, N. J.).

**Leighton, Walter. Bounds for the solutions of a second-order linear differential equation.** Proc. Nat. Acad. Sci. U. S. A. **35**, 190–191, 422 (1949).

The equation $(r(x)y')' + p(x)y = 0$ is considered, where $r$, $p$ and $(rp)'$ are continuous and $r$ and $p$ are positive on $I$: $a \leqq x < \infty$. From the relationship

$$(ry')^2 + rpy^2 = \int_a^x (rp)'y^2dx + C,$$

there follows at once the theorem that if $(rp)' \geqq 0$ on $I$ then every solution $y(x)$ is bounded in $I$, while if $(rp)' \leqq 0$ on $I$ then $r(x)y'(x)$ is bounded on $I$. Corollaries are given.

*N. Levinson* (Cambridge, Mass.).

**Gradštein, I. S. Differential equations with small coefficients for the derivatives and Lyapunov's theory of stability.** Doklady Akad. Nauk SSSR (N.S.) **65**, 789–792 (1949). (Russian)

This review extends the results of a note by the same author [same Doklady (N.S.) **64**, 441–443 (1949); these Rev. **10**, 536] and of papers by A. N. Tihonov [Mat. Sbornik (N.S.) **22(64)**, 193–204 (1948); these Rev. **9**, 588] and by Friedrichs and Wasow [Duke Math. J. **13**, 367–381 (1946); these Rev. **8**, 272]. Let $x, z, f, \cdots$ represent $\mu$-vectors, and $x_i, \cdots$ their components. Let $x(t)$ be a function of $t$ represented by a curve $\Gamma$ in the $(\mu+1)$-space $S$ of $(x_1, \cdots, t)$ and let $U(\epsilon)$ be the union of all the sets in that space given by $t = t_1$, $|x_i - x_i^0(t_1)| < \epsilon$, $i = 1, 2, \cdots, \mu$, for all

$t_1 \in [t_0, T]$. Suppose that $\gamma$ is in the locus (1) $f(x, t)=0$, where the $f_i$ are of class $C^2$ in $U(\epsilon)$ and in a certain $V(\epsilon)$ specified below. Finally assume that along $\gamma$ the characteristic roots of the Jacobian matrix $\|\partial f_i/\partial x_j\|$ have their real parts not exceeding $-2q<0$. Consider now the two associated systems (2) $\eta dx/dt=f(x, t)$, (3) $dz/dt=f(z, t)$, and their solutions on $[t_0, T]$. Here $\eta$ is a small positive parameter, and the initial point for the solution of (2) is to be $x(t_0, \eta)=a(\eta)$. It is assumed finally that for the singular point $x^0(t_0)$ of (3) the trajectory issuing from $a(0)$ is asymptotic to the singular point as $t\to+\infty$. The set $V(\epsilon)$ is now the analogue of $U(\epsilon)$ but for the solution $z(t)$ and $[t_0, t_0+\delta]$. Theorem: under these circumstances the solution $x(t, \eta)$ of (2) satisfies $\lim x(t, \eta)=x^0(t)$, $t_0<t\leqq T$, as $\eta\to+0$.

*S. Lefschetz* (Princeton, N. J.).

**Gradštein, I. S. Nonlinear differential equations with small coefficients for certain derivatives.** Doklady Akad. Nauk SSSR (N.S.) **66**, 789–792 (1949). (Russian)

This note extends the results of two preceding notes by the author [see the preceding review and the first reference cited there]. Let $x, y$ be $m$- and $\mu$-vectors with components $x_i, y_j$ and consider the system (1) $h_i(x, y, t)=0, i=1, 2, \cdots, \mu$, where $h_i$ has continuous first derivatives in all variables and continuous second derivatives in the $y_j$. Let $R$ denote the set of all points $x, y, t$ where the Jacobian matrix $\|\partial h_i/\partial y_j\|$ has all characteristic roots with negative real parts. Consider now the system (2) $dy_i/d\tau=h_i(x, y, t)$ in which $t$ and the $x_i$ are parameters. Clearly every point of $R$ is singular for (2) and more precisely it is a stable generalized focus or node. Let $M^*(x^*, y^*, t^*)$ denote any such point and $H(x^*, t^*)$ the corresponding space $x=x^*$, $t=t^*$. Let also $E(x^*, y^*, t^*)$ denote the set of all points of $H(x^*, t^*)$ such that as $\tau\to\infty$ the integral curve through such a point tends to $y$. Finally write $S=\bigcup E(x^*, y^*, t^*)$. Theorem. Let $\gamma^0(x^0(t), y^0(t))$ be a continuous integral curve of the system (3) $dx_i/dt=f_i(x, y, t)$, $0=h_j(x, y, t)$ issuing from $x^0(t^0)$, $y^0(t^0)$, where the $h_i$ behave as stated above throughout $S$, and the $f_i$ have continuous first partial derivatives. Suppose $\gamma\subset R$ in $[t_0, T]$. Consider also the integral curve $\Gamma(X(t, \eta), Y(t, \eta))$ of $dX_i/dt=f_i(X, Y, t)$, $\eta dY_j/dt=f_j(X, Y, t)$, where $\eta$ is a positive small parameter and $\Gamma$ issues from any point $x^0(t_0)$, $Y_0$, $t_0$ of $E(x^0(t_0), y^0(t_0), t_0)$. Then as $\eta\to 0$ and for $t_0<t\leqq T$, $\Gamma\to\gamma$ coordinate for coordinate. [The papers cited in the preceding review are additional references.]

*S. Lefschetz* (Princeton, N. J.).

**Gradštein, I. S. On the behavior of the solutions of systems of linear differential equations with constant coefficients, degenerating in the limit.** Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 253–280 (1949). (Russian)

Let $\eta$ denote a small parameter. If a function $f(x, \eta)$ tends to a limit as $\eta\to 0$, let this limit be written $f(x)$. Set $p=d/dt$ and consider the two associated systems of $n$ differential equations in $n$ unknowns:

$$(1)\qquad \sum L_{ij}(p, \eta)X_j=\psi_i(t, \eta),$$

$$(2)\qquad \sum L_{ij}(p)x_i=\psi_i(t),$$

where the $L_{ij}$ are polynomials in $p$ and where the degree in $\lambda$ of the determinant $f(\lambda, \eta)=|L_{ij}(\lambda, \eta)|$ is lowered when $\lambda=0$. The author proves a rather intricate sufficiency theorem under which the solutions of (1), under suitable initial conditions, tend to the related solutions of (2) in a bounded interval $0\leqq t\leqq T$. The author asserts that his theorem includes all known results of this nature on nonhomogeneous systems with constant coefficients. He uses the machinery of Laplace transforms throughout. [References: Yü-Why Tschen, Compositio Math. **2**, 378–401 (1935); Andronow and Chaikin, Theory of Oscillations, English language edition, Princeton University Press, 1949, in particular, § 5; these Rev. **10**, 535.] *S. Lefschetz* (Princeton, N. J.).

**LaSalle, J. Relaxation oscillations.** Quart. Appl. Math. **7**, 1–19 (1949).

This paper contains a study of the "van der Pol" equation $\ddot{x}+\mu f(x)\dot{x}+x=0$. Among the many papers on this equation the present contribution is most closely related to the work of Flanders and Stoker [Studies in Nonlinear Vibration Theory, pp. 50–64, Institute for Mathematics and Mechanics, New York University, 1946; these Rev. **8**, 329]. However, instead of considering the special case $f(x)=x^2-1$, more general functions with several oscillations are considered. Ring shaped regions are constructed in the $(x, \dot{x})$-plane in each of which a unique periodic solution is shown to exist. Bounds on the periods and the amplitudes are given, and the asymptotic behavior as $\mu$ tends to $\infty$ is investigated.

*F. Bohnenblust* (Pasadena, Calif.).

**John, Fritz. On harmonic vibrations out of phase with the exciting force.** Communications on Appl. Math. **1**, 341–359 (1948).

A solution of the equation $\ddot{x}=F(x, \cos\omega t)$ is called harmonic and out of phase if it has frequency $\omega$ and assumes its maxima exactly at those values of $t$ at which $\cos\omega t$ has its minima, and conversely. The function $F(x, z)$ is assumed to be of class $C^2$ in the strip $-\infty<x<+\infty$, $-1\leqq z\leqq 1$, and $F_z<0$, $F_x>k>0$, $\lim|F|=\infty$, $\lim F_x/F=0$ uniformly in $z$ as $|x|$ tends to $\infty$. The lower amplitude $a$ of an out of phase harmonic solution is the negative value of the minimum of the solution. It is positive and, given any $a$, there exists exactly one $\omega$ for which the differential equation has an out of phase harmonic solution of lower amplitude $a$. The response curve is the curve in the $(a, \omega)$-plane corresponding to this dependence of $\omega$ on $a$. It has the $\omega$-axis as an asymptotic. The relationship between the stability of the solution and the slope $d\omega/da$ of the response curve is investigated. In particular, if $F$ is symmetric then instability occurs if and only if $da/d\omega>0$ or $=\infty$. The present results are more general and the proofs simpler than those previously presented by the author [Studies in Nonlinear Vibration Theory, pp. 104–192, Institute for Mathematics and Mechanics, New York University, 1946; these Rev. **8**, 329].

*F. Bohnenblust* (Pasadena, Calif.).

**Massera, J. L. The number of subharmonic solutions of non-linear differential equations of the second order.** Ann. of Math. (2) **50**, 118–126 (1949).

The system $\dot{x}=F(x, y, t)$, $\dot{y}=G(x, y, t)$ of differential equations is considered. The functions $F$ and $G$ are periodic with period 1 and such that for any initial conditions $(x_0, y_0, t_0)$ the solution exists for all $t\geqq t_0$ and is unique. A circle and an integer $N$ are assumed to exist such that (1) every solution enters the circle and (2) if at $t_1$ the solution is in the circle then it remains in this circle for all $t\geqq t_1+N$. The number $N(q)$ of subharmonic solutions of least period $q$ is investigated under the assumption that the periodic solutions of rational periods are simple. In a previous paper, N. Levinson [Ann. of Math. (2) **45**, 723–737 (1944); these Rev. **6**, 173] proved that $N(q)$ is an even multiple of $q$ for $q>1$. Massera pointed out that this proof applies only for $q$

odd and that the result must be modified for $q$ even [cf. Levinson, Ann. of Math. (2) **49**, 738 (1948); these Rev. **10**, 457]. The present paper contains the details of this correction and also a study of examples showing that the conditions which are obtained for $N(q)$ are necessary and sufficient. *F. Bohnenblust* (Pasadena, Calif.).

**Cartwright, M. L., and Littlewood, J. E. Addendum to 'On non-linear differential equations of the second order. II.'** Ann. of Math. (2) **50**, 504–505 (1949).

The authors fill in an omission, pointed out by M. H. A. Newman, in their paper [same Ann. (2) **48**, 472–494 (1947); these Rev. **9**, 35]. *N. Levinson* (Cambridge, Mass.).

**Levinson, Norman. A second order differential equation with singular solutions.** Ann. of Math. (2) **50**, 127–153 (1949).

The solutions of the differential equation

$$\epsilon\ddot{x}+f(x)\dot{x}+\epsilon x=b\cos t$$

present, for certain $f$ and $b$, a highly complex behavior; stable subharmonic solutions of different periodicity and discontinuous recurrent motions occur. Cartwright and Littlewood [J. London Math. Soc. **20**, 180–189 (1945); these Rev. **8**, 68] investigated this behavior in the case where $f$ is the simple function $f_0(x)=x^2-1$. Their proofs were only sketched and require a deep analysis. The present paper is devoted to the same question. It deals with the case $f_1(x)=1$ if $|x|>1$ and $f_1(x)=-1$ if $|x|<1$, which qualitatively is similar to the case $f=f_0$ treated by Cartwright and Littlewood. In $|x|>1$ and $|x|<1$ separately the differential equation is linear and thus allows a more direct and explicit formulation of the behavior of the solutions. The author furthermore shows that for any $f$ such that $\int f(x)dx$ approximates $\int f_1(x)dx$ sufficiently closely the solutions of the equation will still behave in the same complex way. In particular, $f$ can be chosen as a polynomial. Thus the author obtains examples of analytic differential equations of the second order whose solutions include discontinuous recurrent motions by a simpler method than the original equation discussed by Cartwright and Littlewood. The main results of this paper had been previously announced [Proc. Nat. Acad. Sci. U. S. A. **34**, 13–15 (1948); these Rev. **9**, 435]. The exact description of the complex behavior of the solutions is too long to be reproduced here. *F. Bohnenblust* (Pasadena, Calif.).

**Levinson, N. On the uniqueness of the potential in a Schrödinger equation for a given asymptotic phase.** Danske Vid. Selsk. Mat.-Fys. Medd. **25**, no. 9, 29 pp. (1949).

The solution of the initial value problem

$$y''+(\lambda^2-P(x))y=0,$$

$y(0,\lambda)=0$, $y'(0,\lambda)=1$, will, under certain assumptions on $P(x)$, and for real $\lambda=u\neq 0$, approach a sine function with phase $\phi(u)$, as $x\to\infty$. The determination of the potential $P(x)$ from the asymptotic phase $\phi(u)$ is an important problem in quantum physics. This method is based on the assumption that $P(x)$ is uniquely determined by $\phi(u)$. The author proves two theorems which establish this uniqueness under very general conditions. (1) If $P(x)$ belongs to the class of functions defined by the conditions (a) $P(x)$ is Lebesgue measurable, (b) $P(x)\geqq 0$ and (c) $\int_0^\infty xP(x)dx<\infty$, then $P(x)$ is the only function in this class having $\phi(u)$ as asymptotic phase. (2) If instead of (b) and (c) one requires (b′) $P(x)$ real and (c′) $\int_0^1 x|P(x)|dx+\int_1^\infty x^2|P(x)|dx<\infty$, then the statement of (1) remains true, provided

$$\phi(\infty)-\phi(+0)<\pi.$$

If the last inequality holds, then there are no discrete characteristic values. The author also sketches extensions of these results to the differential equation

$$y''-x^{-2}l(l+1)y+(\lambda^2-P(x))y=0.$$

One essential part of the proof consists in the derivation of a modified spectral representation formula, which involves in combination solutions of two differential equations with the same $\phi(u)$. This formula is then seen to imply the identity of the two differential equations. *W. Wasow.*

**Levinson, Norman. Determination of the potential from the asymptotic phase.** Physical Rev. (2) **75**, 1445 (1949).

Summary of the paper reviewed above.

**Langer, Rudolph E. On the wave equation with small quantum numbers.** Physical Rev. (2) **75**, 1573–1578 (1949).

Customarily the characteristic values for the one-dimensional wave equation $d^2u/dx^2+\lambda^2\{E-V(x)\}u=0$ are determined asymptotically for large $\lambda$ from

$$\lambda\int_{x_1}^{x_2}\{E-V(x)\}^{\frac{1}{2}}dx=[n+\tfrac{1}{2}]\pi, \tag{1}$$

where $x_1$ and $x_2$ are the two "turning points" of the system, i.e., the two roots of $E-V(x)=0$ and $[n+\frac{1}{2}]$ is equal to $n+\frac{1}{2}$, apart from terms of the order $1/\lambda$. The proof is based upon asymptotic approximations for solutions which are finite everywhere in the $x$-plane. These asymptotic approximations being valid only for large values of the integral occurring in (1), the derivation only holds for large quantum numbers $n$.

The conditions for small quantum numbers are derived for a potential function $V(x)$ which satisfies at a point $x_0$, lying between $x_1$ and $x_2$, the relations $V(x_0)=V'(x_0)=0$ and $V''(x_0)>0$ and further in the neighbourhood of $x_0$ the inequality $V(x)\leqq M(x-x_0)^2$. Since $\lambda E$ has moderate values, the equation can be transformed into a form considered previously by the author [Trans. Amer. Math. Soc. **36**, 90–106 (1934)]. The principal term of an asymptotic expansion, valid in the complete $x$-plane, can be represented by a certain confluent hypergeometric function. From the "crossing substitutions" for this function around the singular point corresponding to $x_0$, analytic expressions for large values of $x$ are found in different regions of the $x$-plane. The condition that these expressions remain finite for all values of $x$ then provides the quantum condition, expressed by (2) $\lambda E=\{2V''(x_0)\}^{\frac{1}{2}}[n+\frac{1}{2}]$.

For functions $V(x)$ satisfying the further condition $M_1(x-x_0)^2\leqq V(x)$ in the neighbourhood of $x_0$, the quantum conditions (1) are shown to yield the same results as (2) if $M$ and $M_1$ are nearly the same, which accounts for the fact, often remarked, that (1) is also valid for small $n$.

The normalization of the characteristic functions is performed according to a method of Furry [Physical Rev. (2) **71**, 360–371 (1947); these Rev. **8**, 463] and the author obtains explicit expressions in terms of the confluent hypergeometric functions. Finally he considers the case of the harmonic oscillator with $V(x)=x^2$, where the first terms of the asymptotic approximations furnish the exact solutions. *J. G. van der Corput* (Amsterdam).

**Manacorda, T. Estensione alle equazioni differenziali lineari del secondo ordine omogenee complete di una formula di Hartmann e Wintner per la valutazione asintotica del numero degli zeri di un integrale.** Boll. Un. Mat. Ital. (3) **3**, 205–210 (1948).

An appraisal for $N(x)$, the number of zeros of a solution of $y''+2a(x)y'+\omega^2(x)y=0$ over $(0, x)$, is obtained when there exists an $l$ such that $0\leqq a(x)<l<\omega(x)$ and when $\omega'(x)=o(\omega^2(x))$. The appraisal yields an asymptotic relationship for the case $a(x)\equiv 0$ due to Hartman and Wintner [Amer. J. Math. **70**, 1–10 (1948); these Rev. **9**, 435].

*N. Levinson* (Cambridge, Mass.).

**Hartman, Philip, and Wintner, Aurel. On the Laplace-Fourier transcendents.** Amer. J. Math. **71**, 367–372 (1949).

The results of Wintner [same J. **69**, 87–98 (1947); these Rev. **8**, 381] are shown to hold for the differential equation

$$x''+x'\sum_{n=0}^{\infty}(-1)^n a_n/t^{2n+1}+x\sum_{n=0}^{\infty}(-1)^n b_n/t^{2n}=0.$$

In the earlier paper the coefficient of $x'$ was zero.

*N. Levinson* (Cambridge, Mass.).

**Wintner, Aurel. On linear repulsive forces.** Amer. J. Math. **71**, 362–366 (1949).

Let $x$ be a vector with $n$ components and let $F(t)$ be a real symmetric matrix of $n$ rows and columns which is continuous for $t\geqq 0$ and nonnegative definite for every fixed $t$. Then the system of differential equations $x''=F(t)x$ has $n$ linearly independent solution vectors $x(t)$ which satisfy $r(t)>0$, $r'(t)\leqq 0$, $r''(t)\geqq 0$ for $t\geqq 0$, where $r(t)=|x(t)|^2$. The result remains valid if $F(t)$ is replaced by $F(t)+S(t)$, where $S(t)$ is skew-symmetric and continuous. The case $n=1$ is due to Kneser [J. Reine Angew. Math. **116**, 178–212 (1896)].

*N. Levinson* (Cambridge, Mass.).

**Timman, R. Asymptotic formulae for special solutions of the hodograph equation in compressible flow.** Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F.46**, i+26 pp. (1949).

This paper is concerned with the approximate solution of the differential equation

$$\frac{d^2W}{d\tau^2}-\left\{\frac{n^2-1}{4}\,\frac{1-\tau/\tau_s}{\tau^2(1-\tau)}+\frac{(1-\tau_s)(1+3\tau_s)}{16(1-\tau)^2\tau_s^2}\right\}W=0,$$

where $n$ is a parameter and $\tau_s$ a constant. This equation is obtained from the hodograph equation for the Legendre potential of the flow of a polytropic gas if the Legendre potential is assumed to be periodic with respect to the direction of the velocity. By an adaptation of the method of R. E. Langer [Trans. Amer. Math. Soc. **34**, 447–480 (1932)], the author derives convergent series representations for two sets of fundamental systems of this differential equation, one set being represented near $\tau=0$, the other near $\tau=\tau_s$. For large values of $n$ these series lead to simple asymptotic formulas for the solutions. The asymptotic formulas for $\tau<\tau_s$ (subsonic range) are different from those for $\tau>\tau_s$ (supersonic range).

*W. Wasow* (Los Angeles, Calif.).

**Magenes, Enrico. Problemi di valori al contorno per l'equazione differenziale $y^{(n)}=\lambda f(x, y, y', \cdots y^{(n-1)})$.** Ann. Mat. Pura Appl. (4) **27**, 39–74 (1948).

The author determines conditions upon $f$ under which there exists a value of $\lambda$ for which the equation of the title, together with the boundary conditions

$$y(x_i)=y_{i0}, \cdots, y^{(c_i-1)}(x_i)=y_{ic_i-1}, \quad 1\leqq i\leqq k,$$
$$\sum c_i=n+1, \quad a\leqq x_i\leqq b,$$

has a solution. This is a nonlinear Sturm-Liouville problem which was first posed by Zawischa for the first order equation. Recently Zawischa's results were considerably improved by Zwirner [Rend. Sem. Mat. Univ. Padova **15**, 33–39 (1946); these Rev. **8**, 206]. The author extends the results of the cited authors using the classical device of converting the differential equation into an integral equation via Green's functions. He applies the method of successive approximations and the fixed-point theorem of Birkhoff-Kellogg-Cacciopoli. Particular attention is paid to the second order equation $y''=f(x, y, y')$.

*R. Bellman.*

**Charles, Henri. Sur une certaine classe d'équations aux différentielles totales. I.** Bull. Soc. Roy. Sci. Liége **18**, 25–30 (1949).

**Charles, Henri. Sur une certaine classe d'équations aux différentielles totales. II.** Bull. Soc. Roy. Sci. Liége **18**, 120–123 (1949).

The object of each paper is to give conditions on the functions $A(x, y, z)$, $B(x, y, z)$ sufficient to insure a solution but not a unique solution of the total differential equation $dz=Adx+Bdy$ that will pass by a given point $(x_1, y_1, z_1)$.

*F. G. Dressel* (Durham, N. C.).

**Franckx, E. L'intégration des systèmes normaux d'équations différentielles et la méthode des approximations successives.** Bull. Soc. Roy. Sci. Liége **17**, 281–286 (1948).

The integration of the equation $y_i'=f_i(x, y_1, \cdots, y_N)$ is often achieved by converting it into the integral equation $y_i=y_i(0)+\int_0^x f_i dx$ and applying the method of successive approximations in some fashion. The author shows that these different procedures may be applied simultaneously and in arbitrary order without hindering the convergence of the sequence, under certain conditions upon the functions $f_i$.

*R. Bellman* (Stanford University, Calif.).

**Franckx, E. La méthode des approximations successives et les systèmes différentiels normaux à une infinité d'inconnues.** Bull. Soc. Roy. Sci. Liége **17**, 308–312 (1948).

The author considers the infinite system of differential equations $y_i'=f_i(x, y)$, $1\leqq i<\infty$, where the right-hand sides satisfy uniform Lipschitz conditions of the form

$$|f_i(x, \bar{y})-f_i(x, y)|\leqq\sum_k a_k|\bar{y}_k-y_k|,$$

$\sum a_k<\infty$. Existence and uniqueness of the initial value problem are demonstrated. [The method and result are known.]

*R. Bellman* (Stanford University, Calif.).

**Germay, R.-H. Sur les systèmes complètement intégrables d'équations aux différentielles totales.** Bull. Soc. Roy. Sci. Liége **17**, 200–208 (1948).

Noting first that the solution of the completely integrable system

$$dz_j=\sum_{k=1}^{n}a_{jk}(x, z)dx_k, \quad a_{jk}(x, z)=a_{jk}(x_1, \cdots, x_n; z_1, \cdots, z_p),$$

$j=1, \cdots, p$, which passes by the point $x=z=0$, can be written in the form

$$(*) \qquad z_j(x)=\int_0^1\sum a_{jk}(tx, z(tx))x_k dt,$$

the paper then presents a method of successive approximations for solving (*) provided the $|x_i|$ are all sufficiently small. *F. G. Dressel* (Durham, N. C.).

**Germay, R.-H. Sur l'intégration par approximations successives de systèmes normaux d'équations différentielles dont les seconds membres sont donnés comme limites de suites uniformément convergentes.** Bull. Soc. Roy. Sci. Liége 16, 247–254 (1947).

The author considers the problem of the integration of the system of differential equations $y_i' = f_i(x, y_1, \cdots, y_N)$, $1 \leqq i \leqq N$, where $f_i$ is the uniform limit as $n \to \infty$ of a sequence $f_{in}$. The usual method of successive approximations is applied to $y' = f_{in}$. *R. Bellman.*

**Germay, R. H. Sur la propriété d'inversion des intégrales de Cauchy d'un système d'équations différentielles de forme normale.** Bull. Soc. Roy. Sci. Liége 18, 32–37 (1949).

Consider the equation $dy_i/dx = f_i(x, y_1, \cdots, y_N)$, $y_i(x_0) = y_i^0$, $1 \leqq i \leqq N$, where the right-hand sides are analytic functions of their arguments within sufficiently small circles surrounding the initial values. For values of $x$ sufficiently close to $x_0$, the solution of the equation may be expressed implicitly as follows: $y_i^0 = g_i(x, y_1, \cdots, y_N)$. The question treated is that of determining an upper bound for the permissible distance $|x - x_0|$. Previous estimates have been given. The method of majorants is used to derive a new estimate. *R. Bellman* (Stanford University, Calif.).

**Germay, R. H. Remarque sur l'emploi d'équations linéaires à coefficients variables dans la méthode d'intégration par approximations successives des équations différentielles normales.** Bull. Soc. Roy. Sci. Liége 18, 3–8 (1949).

The author points out that when the right-hand side of the equation $dz/dx = F(z, x)$ has the form $a(x)z + G(z, x)$, better results may be obtained from the method of successive approximations applied in the fashion

$$dz_{n+1}/dx = a(x)z_{n+1} + G(z_n, x)$$

than from the method applied in the straightforward manner. [This idea is well known and forms an essential part of the stability theory of nonlinear differential equations.] *R. Bellman* (Stanford University, Calif.).

**Dehousse, L. Remarques sur les approximations successives relatives à un système linéaire d'équations différentielles.** Bull. Soc. Roy. Sci. Liége 16, 162–173 (1947).

Referring to the results described in the preceding review, the author makes precise the expression "better results" by showing that, if $a(x) > 0$, the convergence will be more rapid than that of the ordinary method. The result is generalized to apply to systems where the elements of the main diagonal of the coefficient matrix are positive. *R. Bellman.*

**Germay, R. H. J. Sur l'intégrale de Darboux d'une équation aux dérivées partielles du premier ordre.** Bull. Soc. Roy. Sci. Liége 18, 102–107 (1949).

The author gives a method for solving the first order differential equation (*) $F(x_1, \cdots, x_n; z; p_1, \cdots, p_n) = 0$ without first putting (*) in solved form

$$p_1 = f(x_1, \cdots, x_n; z; p_2, \cdots, p_n).$$

The present paper generalizes one of the methods previously sketched by the author for solving (*) [Ann. Soc. Sci. Bruxelles. Sér. I. 61, 99–105 (1947); these Rev. 8, 584]. *F. G. Dressel* (Durham, N. C.).

**Konakov, P. K. On the second theorem of similarity.** Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1949, 240–248 (1949). (Russian)

Le second théorème de similitude (théorème des II ou théorème de Buckingham) relatif à une équation finie, peut-être envisagé comme cas particulier d'une proposition de A. Federman [cf. Ann. [Izvestia] Inst. Polytech. St. Petersbourg 16, 97–155 (1911)]. Ehrenfest-Afanassjewa [Math. Ann. 77, 259–276 (1916); Philos. Mag. (7) 1, 257–272 (1926)] a étendu ce théorème au cas ou la relation entre les paramètres considérés contient, en outre, des valeurs particulières des grandeurs physiques envisagées. L'auteur fait à la démonstration précédente des objections qu'il n'explicite pas. Dès lors, il se propose d'indiquer une proposition encore plus générale dont les énoncés usuels ne seraient que des cas particuliers.

Soient: $(E)$ une relation différentielle liant $n$ variables $x_i$ dans un domaine $D$ limité par une surface fermée $S$; $x_{0i}$ les valeurs numériques de ces variables connues en certains points de l'ensemble de fermeture $V+S$. Alors, l'intégrale générale de $(E)$ relativement au groupe de transformations continus $G$ de $(E)$ se présente sous forme d'une relation entre les invariantes de $G$, définie d'une manière univoque lorsqu'on connaît une relation invariante entre les $x_{0i}$. L'auteur identifie cette proposition à celle de Buckingham dans le cas ou $G$ est le groupe des similitudes. *J. Kravtchenko* (Grenoble).

**Gião, Antonio. Le problème général aux limites pour les fonctions continues spatio-temporelles et les équations intégrales de l'hydrodynamique.** C. R. Acad. Sci. Paris 228, 1275–1276 (1949).

Let $V^m$ $(m = 1, 2, 3)$ satisfy the equations

$$(*) \qquad \partial V^m/\partial t - K\Delta V^m = A^m(x_1, x_2, x_3, t),$$

where $\Delta$ is the Laplacian operator and $K$ any constant. Writing

$$\Phi(r, t, \tau) = (t-\tau)^{-\frac{1}{2}} \exp\left[-r^2/4K(t-\tau)\right], \quad \Psi = r^{-1}\int_{r'}^{r} \Phi dr,$$

where $r$ is the distance between two points and $r'$ any constant, the author follows Oseen in defining the tensor and vector

$$\bar{W}_i^k = -\delta_i^k \Delta\Psi + \Psi_{,ik}, \quad \bar{B}^k = (\partial\Psi/\partial\tau + K\Delta\Psi)_{,k},$$

so that $\partial\bar{W}_i^k/\partial\tau + K\Delta\bar{W}_i^k = \bar{B}_{,i}^k$, $\bar{W}_{k,i}^i = 0$. The limit $W_i^k$ of $\bar{W}_i^k$ as $r'$ tends to zero is used as kernel in the central integral formula of the paper. This formula is stated without details of calculation, which are to appear elsewhere. It expresses the partial derivatives $V_{,i}^m$ at time $t_1$ at any point $P$ in a domain $D(t_1)$ in terms of three integrals taken respectively over the domain $D(t_0)$, over the history from $t = t_0$ to $t = t_1$ of the surface $S(t)$ which bounds $D(t)$, and over $S(t_1)$. The integrands involve $W_i^k$ and $V^m$ and their partial derivatives, and also $A^m$, this last occurring on $S(t)$ only. By virtue of (*), $A^m$ is there expressible in terms of partial derivatives of $V^m$. Application to hydrodynamics is indicated, (*) being the partial differential equation of motion of a compressible viscous fluid. [It appears strange to the reviewer that it should be possible to express the solution of (*), or more precisely the partial derivatives $V_{,i}^m$, in terms of the boundary values of $A^m$ on $S(t)$, without reference to the values of $A^m$ in $D(t)$. Or, to put it another way, if we regard (*) not as a partial differential equation but as the definition of $A^m$, the author's formula expresses $V^m$ in terms of boundary

values without $V^m$'s being subjected to any partial differential equation at all.] *J. L. Synge* (Dublin).

**Sauer, R. Bemerkungen zur Charakteristikentheorie der partiellen Differentialgleichungen 2. Ordnung.** Z. Angew. Math. Mech. **25/27**, 151–153 (1947).

The linear differential equation of the second order which is homogeneous in the second derivatives is transformed by the Legendre transformation into a nonlinear equation again homogeneous in the second derivatives. In the hyperbolic region the two characteristic nets are geometrically related, one being reciprocally orthogonal to the other. In particular, if one net consists only of straight lines, the other net is a parallelogram-net consisting of two families of congruent curves. In these cases the general solutions of the differential equations are given explicitly. The analogous solutions in the elliptic region are also given. The general theory is directly applicable to many problems of differential geometry or compressible fluid flow which are integrable in closed form, of which three are given as examples. These are: (a) Weierstrass's representation of the minimal surface; (b) linearized conical supersonic flow, investigated by Busemann and others; and (c) nonlinear plane pressure waves in the cases $\gamma=-1$ and $\gamma=3$.

*W. D. Hayes* (Providence, R. I.).

**Oleĭnik, O. A. On the Dirichlet problem for equations of elliptic type.** Mat. Sbornik N.S. **24(66)**, 3–14 (1949). (Russian)

L'auteur considère l'équation aux dérivées partielles du type elliptique:

$$(1)\qquad \sum_{i=1}^{N}\sum_{k=1}^{N} a_{ik}\frac{\partial^2 u}{\partial x_i \partial x_k}+\sum_{1}^{N} b_i\frac{\partial u}{\partial x_i}+cu=0,$$

les $a_{ik}$ (respectivement $b_i$ et $c$) admettant des dérivées partielles troisièmes (respectivement secondes et premières) continues au sens de Hölder (sauf pour les $\partial c/\partial x_i$ qu'on suppose simplement continues) dans le domaine $G$ de l'espace à $N$ dimensions; de plus, $c\leqq 0$ dans $G$. Soient $D$ un domaine complètement intérieur à $G$, $S$ la frontière de $D$, $f(P)$ une fonction continue du point $P\varepsilon S$; $u_f(M)$ ($M\varepsilon D$) la solution généralisée (au sens de Wiener) du problème de Dirichlet posé relativement à (1) dans $D$ avec la donnée frontière $f(P)$. Le point $A\varepsilon S$ est dit alors regulier, si, quel que soit $f(P)$, on a: $\lim_{M\to A} u_f(M)=f(A)$.

G. Tautz [Math. Ann. **117**, 694–726 (1941); **118**, 733–770 (1941); ces Rev. **3**, 126; **6**, 3] a prouvé que pour $N=2$ un point $A\varepsilon S$, régulier pour (1), l'est encore pour l'équation de Laplace et réciproquement; pour $N>2$, Tautz démontre seulement le théorème direct et encore pour certaines formes de (1). L'auteur comble cette lacune de la théorie et établit la critère de régularité ci-dessus dans toute sa généralité pour toutes les équations du type (1); à la fin de son travail, il parvient même à se débarasser de la restriction $c\leqq 0$. Enfin, l'auteur note que si le problème de Dirichlet est résoluble pour une équation particulière (1), dans $D$, seulement pour des fonctions particulières $f(P)$ ($P\varepsilon S$), ce même problème peut n'être pas résoluble avec les données $[D, f(P)]$ pour d'autres équations (1); c'est ce que prouve un exemple simple.

A noter que W. Püschel [cf. Math. Z. **34**, 535–553 (1932)] avait complètement justifié le théorème ci-dessus pour des équations du type:

$$\sum_{k=1}^{N}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right)+cu=0,$$

la forme $\sum\sum a_{ik}X_iX_k$ étant encore définie positive.

*J. Kravtchenko* (Grenoble).

**Riesz, Marcel. L'intégrale de Riemann-Liouville et le problème de Cauchy.** Acta Math. **81**, 1–223 (1949).

The author has developed a theory of multi-dimensional fractional integration and has derived from it a powerful method for the solution of linear differential equations of the hyperbolic type [C. R. Congrès Internat. Math. Oslo 1936, v. 2, pp. 44–45 (1937); and several other publications by the author and by N. E. Fremberg; cf. also B. B. Baker and E. T. Copson, The Mathematical Theory of Huyghens' Principle, Oxford University Press, 1939; these Rev. **1**, 315]. The present paper contains an account of all results previously published without detailed proofs. Formally, therefore, we have a research paper. However, the paper is written in the style of a book. All details are given and, where desirable, passages of previously published papers are reproduced. The author takes care to compare his method with others and to point out the details which make the mechanism work. The paper is self-contained and should be accessible also to non-experts.

As an introduction and survey of the results, a lecture of 1937 is reprinted. For a general orientation, chapter 1 describes the theory of Riemann-Liouville integrals

$$(1)\qquad I^\alpha f(x)=\{1/\Gamma(\alpha)\}\int_a^x f(t)(x-t)^{\alpha-1}dt.$$

For $\alpha>0$ the integral defines an analytic function of $\alpha$. If $f(x)$ has $n$ derivatives, then $n$ integrations by parts reduce $I^\alpha f(x)$ to a polynomial in $x$ plus $I^{\alpha+n}f^{(n)}(x)$. The latter integral converges for $\alpha>-n$, and thus $I^\alpha f(x)$ is defined by analytic continuation for all $\alpha>-n$. For negative integers $m$ we have $I^{-m}f(x)=f^{(m)}(x)$ but for no other value of $\alpha$ is the operator $I^\alpha$ of a local character. This phenomenon is due to the vanishing of $1/\Gamma(\alpha)$ for $\alpha=-m$ and, as the author points out, the Huyghens principle is only a counterpart of it. The operator $I^\alpha$ satisfies the semi-group property $I^\alpha(I^\beta)=I^{\alpha+\beta}$.

In chapter 2 the theory is generalized to $m$-dimensional Euclidean space where an operator

$$(2)\qquad I^\alpha f(P)=\{1/H_m(\alpha)\}\int f(Q)r_{PQ}^{\alpha-m}dQ$$

has similar properties; here $r_{PQ}$ is the distance between the points $P$ and $Q$ and the integral extends over the entire space. The integral converges for $\alpha>m$, but for sufficiently regular $f(Q)$ Green's formula leads to

$$I^\alpha f(P)=(-1)^p I^{\alpha+2p}\Delta^p f(P),$$

so that analytic continuation defines $I^\alpha$ for all $\alpha$. In particular, $\Delta I^2 f(P)=-f(P)$ so that $I^2$ is the inverse of $-\Delta$; moreover, $I^0 f(P)=f(P)$.

The main part of the paper begins with chapter 3 where the operator (2) is defined for a Lorentzian space with metric $r_{PQ}^2=(x_1-\xi_1)^2-(x_2-\xi_2)^2-\cdots-(x_m-\xi_m)^2$. The integration now extends over the interior of the retrograde cone $C^P$ with vertex at $P=(x_1,\cdots,x_m)$ defined by $r_{PQ}=0$, $x_1-\xi_1>0$. The operator $\Delta$ is now

$$\partial^2/\partial x_1^2-\partial^2/\partial x_2^2-\cdots-\partial^2/\partial x_m^2.$$

The formal theory goes as before, but when Green's formula is applied we get surface integrals on $C^p$ in which the singularity is of the same relative order as in $I^\alpha$ so that no improvement is achieved. Accordingly, the analytic continuation can not be effected as simply as before. The new method of effective continuation is an improvement of the methods of N. E. Fremberg [Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **15**, no. 27, 265–276 (1945); Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] **7**, 1–100 (1946); these Rev. **7**, 384; **8**, 384].

Suppose now that $S$ is an $(m-1)$-dimensional surface with "space orientation," and denote by $S^p$ the bounded domain on $S$ which lies inside the cone $C^p$. Similarly, let $C_S{}^p$ be the bounded portion of $C^p$ "above" $S$. The two surfaces $C_S{}^p$ and $S^p$ together form the boundary of a bounded portion $D_S{}^p$ of the interior of $C^p$. Define $f(Q)$ to be zero outside $D_S{}^p$ and apply Green's formula formally to (2). We get $I^\alpha f(P)$ as a sum of $I^{\alpha+2}\Delta f(P)$ and two surface integrals on $S^p$ which involve $f$ and its first derivatives. The integrals diverge for $\alpha=0$ and receive a meaning only by the delicate process of analytic continuation. However, once Green's formula is justified, we have a formal solution of the Cauchy problem for the wave equation $\Delta f=h$. In fact $I^0 f(P)=f(P)$ is then expressed in terms of the space integral $I^2 h$ and two integrals involving the given boundary values of $f$ and its first derivatives along $S$. It is not difficult to justify this formal solution. Hadamard obtained a solution by defining the "finite part" of certain divergent integrals. The same end is here achieved by analytic continuation which avoids certain difficulties and, in particular, does not distinguish between even and odd numbers of dimensions. These developments are given in chapter 4 together with a detailed analysis of the Huyghens principle and boundary value problems for the direct cone, time-oriented surfaces, etc.

Chapter 5 contains a purely geometric interpretation of the solution for the physically most important case, namely $m=4$, with a discussion of certain line congruences and caustics. The entire discussion is of an invariant character. Chapter 6 applies the method to the Maxwell and Dirac equations and discusses the Liénard-Wiechert potential of a moving electron [cf. also Fremberg, Proc. Roy. Soc. London. Ser. A. **188**, 18–31 (1946); these Rev. **8**, 302]. Finally, chapter 7 generalizes the theory to the wave equation in arbitrary Riemann spaces. The fundamental operator now involves a function of the form

$$S^{\alpha-m}\sum_{k=0}^{\infty} V_k S^{2k}/H_m(\alpha, k),$$

where $S$ is the geodesic distance and the $V_k$ are obtained from certain differential equations. Unfortunately limitations of space do not permit a more detailed exposition of the results of the last three chapters. *W. Feller.*

**Atkinson, F. V. On Sommerfeld's "radiation condition."** Philos. Mag. (7) **40**, 645–651 (1949).

The author proves the unique determination of a solution of the wave equation outside a surface $S$ by its boundary values on $S$, if the Sommerfeld radiation condition and finiteness condition are satisfied at infinity. Let $u(x, y, z)$ be a twice continuously differentiable solution of the equation $\Delta u+k^2u=0$ on and outside the closed surface $S$, where $k$ is a nonvanishing complex constant. Let in addition (1) $u=0$ on $S$, (2) $ru$ be bounded at infinity, (3) $\lim_{r\to\infty} r(iku-u/r)=0$ uniformly in all directions, where $r$ denotes the distance from the origin. Then $u\equiv 0$. The proof is based on the series expansion of $e^{-ikr}u$ according to powers of $1/r$. A proof of this theorem for the case of real positive $k$ has been given by F. Rellich [Jber. Deutsch. Math. Verein. **53**, 57–65 (1943); these Rev. **8**, 204] using conditions (1) and (3) only. *F. John* (New York, N. Y.)

**Friedlander, F. G. Note on the geometrical optics of diffracted wave fronts.** Proc. Cambridge Philos. Soc. **45**, 395–404 (1949).

It is well known that a solution of the wave equation $\nabla^2 u=\partial^2 u/\partial t^2$ in any number of dimensions, for which $u$ and $\partial u/\partial t$ vanish initially outside a closed surface $S_0$, vanishes at time $t$ in the exterior of a surface $S_t$ parallel to and at a normal distance $t$ from $S_0$. The present paper deals with the extension of these properties to problems involving reflexion and diffraction, where there is a shadow. For simplicity, the case of wave propagation in a plane is considered first. The waves are reflected at certain curves $\sigma$ on which $u$ or its normal derivative vanish for all time; the region outside $\sigma$ is denoted by $D$. Initially $u$ and $\partial u/\partial t$ vanish outside a certain closed curve $C_0$. If $D$ were the whole plane, the rays would be the normals to $C_0$ and the wave fronts the orthogonal trajectories of the rays. But it is shown that, in a shadow, the wave fronts are the involutes of $\sigma$, the rays the tangents of $\sigma$; and the wave front is propagated along $\sigma$ with unit velocity. From this it follows that the arrival time at any point, no matter whether in a shadow or not, is given by Fermat's principle, being the least distance from points of $C_0$, measured along a path lying in $D$. The result is then extended to the generalised wave equation

$$\frac{\partial^2 u}{\partial t^2}=\sum_{i,k}\frac{\partial}{\partial x_i}\left(A_{ik}\frac{\partial u}{\partial x_k}\right)+\sum_i B_i\frac{\partial u}{\partial x_i}+Gu,$$

where $A_{ik}, B_i, G$ are functions of the coordinates $(x_1, \cdots, x_m)$ of a point in a space of $m$ dimensions. It is shown that Fermat's principle still holds, the arrival time at any point being equal to the least geodesic distance from the initial wave front, measured along a path in $D$, provided that the metric of space is taken to be $ds^2=\sum a_{ik}dx_i dx_k$, where $a_{ik}$ and $A_{ik}$ are corresponding elements in reciprocal matrices. In the course of this work, a new approach to the theory of characteristics is given. *E. T. Copson* (Dundee).

**Giuliano, Salvatore. Sulle soluzioni comuni a due equazioni di Laplace del tipo iperbolico lineari ed a coefficienti variabili.** Matematiche, Catania **2**, 37–57 (1947).

On déduit, par des considerations élémentaires, les conditions d'intégrabilité du système

$$\frac{\partial^2\theta}{\partial u\partial v}=a\frac{\partial\theta}{\partial u}+b\frac{\partial\theta}{\partial v},\quad \frac{\partial^2\theta}{\partial u\partial v}=a_1\frac{\partial\theta}{\partial u}+b_1\frac{\partial\theta}{\partial v}.$$

On calcule ensuite la solution dans le cas des coefficients constants. On applique les conclusions pour calculer des solutions particulières de l'équation de Laplace

$$\frac{\partial^2\theta}{\partial u\partial v}=a\frac{\partial\theta}{\partial u}+b\frac{\partial\theta}{\partial v}$$

à coefficients constants. *B. Levi* (Rosario).

### *Difference Equations, Special Functional Equations*

**Bellman, Richard. On the existence and boundedness of solutions of non-linear differential-difference equations.** Ann. of Math. (2) **50**, 347–355 (1949).

The existence and boundedness of solutions of

$$(*) \qquad u'(t+1)=a_1u(t)+a_2u(t+1)+f(u(t), u(t+1))$$

are considered, where $u'=du/dt$, $a_1$ and $a_2$ are constants and the coefficients $b_1(t)$ and $b_2(t)$ of the linear terms $u(t)$ and $u(t+1)$ of $f$, tend to zero as $t\to\infty$. The approach is the standard one, namely to replace (*) by

$$u(t)=u_0(t)-\int_0^t k(t-\tau)f d\tau$$

where $f=f(u(\tau), u(\tau+1))$, i.e., the analogue of the "variation of constants" formula in the case of the ordinary differential equation. For stability it is assumed that the roots of $a_2+a_1e^s-se^s=0$ lie to the left of $\Re(s)=-\sigma_0<0$. (This is the characteristic equation in the case where $f\equiv 0$ in (*).) Extensions to systems are pointed out.

*N. Levinson* (Cambridge, Mass.).

**Bruwier, L. Sur l'application du calcul symbolique à la résolution d'équations fonctionnelles.** Bull. Soc. Roy. Sci. Liége **17**, 230–245 (1948).

The author solves the system of functional equations

$$\frac{d}{dt}x_r(t)+\sum_{s=1}^{n}a_{rs}x_s(jt)=f_r(t), \quad r=1, 2, \cdots, n,$$

by operational methods. The $a_{rs}$ are constants, $j$ is a root of unity, and the $f_r(t)$ are entire functions of the class $L^0$ [Doetsch, Theorie und Anwendung der Laplace-Transformation, Springer, Berlin, 1937]. *A. Erdélyi.*

**Bruwier, L. Sur l'application du calcul symbolique à l'intégration d'une équation différentielle fonctionnelle.** Bull. Soc. Roy. Sci. Liége **18**, 72–82 (1949).

[Cf. the preceding review.] In the present paper the author solves

$$\frac{d}{dt}x(t)+ax(jt)=\varphi(t), \qquad a\neq 0,$$

and

$$\frac{d}{dt}x(t)+\sum_{r=0}^{m-1}a_rx(j^rt)=\varphi(t),$$

where $j$ is a primitive $m$th root of unity. *A. Erdélyi.*

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### *Integral Equations*

**Langenhop, Carl Eric. Properties of kernels of integral equations whose iterates satisfy linear relations.** Iowa State Coll. J. Sci. **23**, 50–52 (1948).

The paper is a summary of the author's doctoral thesis. The principal result stated is that if $K(x, y)$ is a bounded measurable kernel and can be expressed as a finite linear combination of its iterates $K_n(x, y)$ for $n\geqq 2$, then it is necessarily of the form $\sum_{i=1}^{N}u_i(x)v_i(y)$. Some results on polynomial Fredholm determinants are then stated. The paper concludes with a discussion of idempotent kernels, i.e., kernels such that $K(x, y)=K_2(x, y)$.

*F. Smithies* (Cambridge, England).

**Kreisel, G. Some remarks on integral equations with kernels: $L(\xi_1-x_1, \cdots, \xi_n-x_n; \alpha)$.** Proc. Roy. Soc. London. Ser. A. **197**, 160–183 (1949).

The paper deals with integral equations of the type $k(\xi)=\int_{-\infty}^{\infty}\psi(x)L(\xi-x)dx$, where $\xi, x$ are $n$-dimensional vectors, $L$ is absolutely integrable, $k(\xi)$ is a given bounded function, and only bounded solutions $\psi(x)$ are considered. It is shown that, under certain conditions of a type familiar in the theory of Fourier transforms, it is possible to write

$$\bar{\psi}(x)=\int_{-\infty}^{\infty}\psi(u)\mu(x-u)du=\int_{-\infty}^{\infty}k(\xi)\lambda(x-\xi)d\xi,$$

where $\lambda(x)$ is integrable in $(-\infty, \infty)$. Under certain extra conditions on the behaviour of $L(\zeta)$ for complex $\zeta$, this expression for the smoothed function $\bar{\psi}(x)$ is used to provide estimates for the lower bound of $\max|\psi|$ in terms of $k$. The author indicates applications of these results to theories of gravity surveying and the propagation of waves in dispersive media. *H. R. Pitt* (Belfast).

**Parodi, Maurice. Sur un type d'équations intégrales de seconde espèce résolubles par le calcul symbolique.** J. Math. Pures Appl. (9) **28**, 35–62 (1949).

Most of the material of this paper was presented in a series of notes [C. R. Acad. Sci. Paris **226**, 43–45, 153–155, 1237–1239, 1877–1878 (1948); **227**, 810–812 (1949); these Rev. **9**, 189, 357, 593; **10**, 36, 370; some of the references seem to be misprinted in the paper]. In the present version some of the results are elaborated, but conditions of validity and rigorous proofs are still lacking. *A. Erdélyi.*

**Tortorici, Paolo. Soluzione approssimata di un'equazione integrale di Cantelli.** Ann. Mat. Pura Appl. (4) **27**, 75–86 = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **232** (1948).

F. P. Cantelli [Atti Accad. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (5) **12**, 397–411 (1919)] posed the following question. Let $X$ be a normal stochastic variable. Does there exist a normal variable $Y$ with a mean (not constant) depending on the chosen value of $X$, in such a way that the variable $X+Y$ is normal? The question is translated into the solution of the integral equation

$$(1) \qquad F(\alpha)=(2\pi\mu)^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{\frac{1}{2\mu}[f(x)-\bar{f}-\mu]^2+\alpha x-\frac{1}{2\mu}x^2}dx-1=0,$$

where

$$(2) \qquad \bar{f}=(2\pi\mu)^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{-\frac{1}{2\mu}x^2}f(x)dx.$$

This equation has the evident solution $f(x)\equiv\bar{f}=$constant. The answer to the question will be affirmative if there exists a solution $f(x)\neq\bar{f}$ for every arbitrary real value of $\alpha$. The author has examined possible solutions of (1) from the class of quasi-continuous functions bounded in $(-\infty, \infty)$ and belonging to $L^2$. He proves (I) every solution of (1) satisfies (2); (II) for a fixed arbitrary $\mu$, $F(\alpha)$ is an integral transcendental function of $\alpha$; (III) (1) is equivalent to the system of $\infty^1$ integral equations (3) $F^{(s)}(0)=0$ $(s=0, 1, 2, \cdots)$ $(F(0)=F'(0)=0$ is obvious); (IV) $f(x)-\bar{f}$ can be developed in a series of Hermite functions $f(x)-\bar{f}\simeq\sum_{k=0}^{\infty}c_k\omega_k(x)$ $(\omega_k(x)=e^{-\frac{1}{2}x^2}(-1)^ke^{x^2}d^ke^{-x^2}/dx^k)$, which converges in mean to $f(x)-\bar{f}$. In $\varphi_p(x)=\sum_{k=0}^{p}c_k\omega_k(x)$, the $p+1$ constants $c_0, c_1, \cdots, c_p$ can be determined in such a way that (3) is satisfied for $s=2, 3, \cdots, p+2$. The author gives the expansions for $p\leqq 4$ and remarks that the integral norms $N[\varphi_p]=\int_{-\infty}^{\infty}[\varphi_p(t)]^2dt$ decrease very rapidly with increasing $p$. The question of whether or not $\lim_{p\to\infty}N[\varphi_p]=0$ (i.e.,

$f(x)=\bar{f}$ is the only solution of (1)) is left unanswered. The article ends with a numerical verification of the approximation obtained by $\varphi_4$. *S. C. van Veen* (Delft).

**Hopf, Eberhard. A mathematical example displaying features of turbulence.** Communications on Appl. Math. 1, 303–322 (1948).

The author studies the following system of nonlinear differential equations with a view similar to that of J. M. Burgers in establishing a model of turbulence:

$$(1)\qquad \begin{aligned} \partial u/\partial t &= -z\circ z^* - u\circ 1 + \mu\partial^2 u/\partial x^2,\\ \partial z/\partial t &= z\circ u^* + z\circ F^* + \mu\partial^2 z/\partial x^2, \end{aligned}$$

where $u(x, t)$ and $z(x, t)$ are the two unknown complex-valued functions of the time $t$ and the angular space variable $x$ mod $2\pi$, $\mu>0$ is a parameter, and $F(x)=a(x)+ib(x)$ is a given complex-valued even function of $x$, absolutely integrable in $(0, 2\pi)$. An asterisk denotes the complex conjugate, and $f\circ g$ is an abbreviation for $(2\pi)^{-1}\int_0^{2\pi} f(x+y)g(y)dy$. Many properties of the solutions of (1) can be obtained. It is shown that there are an infinite number of critical values $\mu_1>\mu_2>\cdots\to 0$ for $\mu$, depending on the Fourier coefficients of $a(x)$ and having the following properties. For $\mu>\mu_1$, there is a stationary "laminar" solution which will be approached by any other solution for the same value of $\mu$. For $\mu_n>\mu>\mu_{n+1}$, there is an $n$-dimensional manifold of "turbulent solutions" such that "almost" all other solutions will approach some of the turbulent solutions with the same value of $\mu$. Other developments include a statistical theory. The ergodic property and the normality of distributions are proved.

The analysis is simplified by the fact that the various spatial Fourier components can be treated separately. The author points out that this lack of interaction between the Fourier components is in contrast to the hydrodynamical problem and suggests that these might perhaps be a starting point in devising and discussing simple models with slight interaction from a continued study of the kind of models proposed. *C. C. Lin* (Cambridge, Mass.).

**Holte, Gunnar. On a method of calculating the density of neutrons emitted from a point source in an infinite medium.** Ark. Mat. Astr. Fys. **35A**, no. 36, 9 pp. (1948).

The formula given by I. Waller [same Ark. **34A**, no. 3 (1947), equation (23); these Rev. **8**, 587] for the space-energy distribution of neutrons in an infinite medium arising from the slowing down by elastic scattering of monoenergetic neutrons emitted by a point source is explicitly evaluated to an order of approximation which is equivalent to including terms up to the third order in an expansion in spherical harmonics. Waller's formula is valid when the mean free path is independent of energy. The case when it is variable has been considered by the author in an earlier paper [same vol., no. 2 (1948); these Rev. **9**, 593]; the author now obtains the solution in an approximation which is higher than that given in that paper.

*S. Chandrasekhar* (Williams Bay, Wis.).

**Agostinelli, C. Sulla integrazione dell'equazione integro-differenziale che regge il fenomeno della diffusione dei neutroni termici.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 140–143 (1948).

The author considers the time dependent equation of transfer

$$(*)\qquad \frac{\partial f(x,\mu,t)}{\partial t}+v\mu\frac{\partial f(x,\mu,t)}{\partial x}=\tfrac{1}{2}\omega_0\int_{-1}^{1}(1+3P\mu\mu')f(x,\mu',t)d\mu' -\omega f(x,\mu,t),\qquad 0\leqq x\leqq\infty,$$

where $v$, $P$, $\omega_0$ and $\omega$ are constant, and shows that the equation admits solutions of the form

$$f(x,\mu,t)=e^{-\omega t}[\Phi(x-vt\mu,\mu)+\Psi(x,t)],$$

where

$$\Phi=G(\mu)\exp\left\{-\frac{3P\omega_0\mu}{v(3P\mu^2+1)}(x-vt\mu)\right\}+\frac{v}{3P\omega_0\mu}H(\mu)$$

and

$$\Psi=\tfrac{1}{2}e^{\omega t}\int_{-1}^{1}(1+3P\mu^2)e^{-\omega t/(3P\mu^2+1)} \times\left[1-\exp\left\{\frac{3P\omega_0\mu}{v(3P\mu^2+1)}x\right\}\right]G(\mu)d\mu,$$

where $G(\mu)$ and $H(\mu)$ are arbitrary functions of $\mu$.

The author states that $G(\mu)$ and $H(\mu)$ can be chosen such that the solution of (*) which satisfies the boundary condition $f(x, \mu, t)=0$ for $x=0$ and $\mu\leqq 0$ can be found.

*S. Chandrasekhar* (Williams Bay, Wis.).

**Pignedoli, Antonio. Sulla teoria della diffusione dei neutroni "termici."** Atti Sem. Mat. Fis. Univ. Modena **2**, 96–107 (1948).

The author treats the same equation as the one considered in the preceding review. The method used is, however, different. Expressing the solution in the form

$$f(x,\mu,t)=e^{-\omega t}F(x,\mu,t),$$

the author applies the double Laplace transformation

$$\Phi(p,q,\mu)=\int_0^\infty e^{-pt}dt\int_0^\infty e^{-qx}F(x,\mu,t)dx,$$

and seeks a solution of the resulting equation by expanding the various terms in inverse powers of $\phi$ and $q$: thus,

$$\Phi(p,q,\mu)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Phi_{m,n}(\mu)p^{-m}q^{-n}.$$

Formulae are given for determining the coefficients $\Phi_{m,n}$ for the case when $f(x, \mu, t)=0$ for $x=0$ and $\mu\leqq 0$.

*S. Chandrasekhar* (Williams Bay, Wis.).

**Charles, Henri. Sur une certaine classe d'équations intégro-différentielles.** Bull. Soc. Roy. Sci. Liége **16**, 291–295 (1947).

Under the hypotheses that $K(x, s)$ is continuous on $x_0\leqq x\leqq x_0+a$, $x_0\leqq s\leqq x_0+a$ and $f(x, y)$, $\Phi(x, y)$ are continuous on $x_0\leqq x\leqq x_0+a$, $|y-y_0|\leqq b$, the author shows by a well-known approximation method that there is an interval $x_0\leqq x\leqq x_0+h$ on which

$$y'(x)=f[x,y(x)]+\int_{x_0}^{x}K(x,s)\Phi[s,y(s)]ds,\qquad y(x_0)=y_0,$$

has a solution; an example is given to show that the above conditions do not imply uniqueness. *W. T. Reid.*

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## *Functional Analysis, Ergodic Theory*

**Fort, M. K., Jr. A unified theory of semi-continuity.** Duke Math. J. **16**, 237–246 (1949).

Let $M$ be a metric space with distance $\rho(X, Y)$, let $P$ be the set of positive real numbers, let o be one of the relations $>$, $<$ and let $F$ be a map of $P\times M$ into a set with a partial order relation $\sim$. Then (o, $F$, $\sim$) is called a continuity structure for $M$ if for all $h, k\in P$ and $X, Y\in M$ the

conditions $h \circ k$ and $\rho(X, Y) < |h-k|$ imply $F(h, X) \sim F(k, Y)$. A map $f$ of a topological space $A$ into $M$ is called (o, $F$, $\sim$)-continuous at $x \varepsilon A$ if $h \circ k$ implies the existence of a neighborhood $V(x)$ such that $F(h, f(x)) \sim F(k, f(y))$ for all $y \varepsilon V$. Ordinary continuity of $f$ implies (o, $F$, $\sim$)-continuity for all (o, $F$, $\sim$), and is equivalent to ($>$, $F$, $\supset$)-continuity where $F(h, X) = \{Y \varepsilon M \mid \rho(X, Y) < h\}$. If $M$ is the reals $R$ metrized by $|X-Y|$, ordinary upper and lower semicontinuity are respectively equivalent to ($>$, $+$, $\geqq$)-continuity and its dual ($<$, $+$, $\leqq$)-continuity, where $+$ denotes the map defined by $F(h, X) = h+X$. Also, the upper and lower semicontinuous functions (whose range is the set of compact subsets of a metric space) which result from semicontinuous decompositions are further examples of dual continuity structures. Preservation of (o, $F$, $\sim$)-continuity is proved for a uniformly convergent sequence $f_i$. Other results include the theorem that if $f$ is (o, $F$, $\sim$)-continuous on $A$ into $M$ and if $f(A)$ is totally bounded, then the set of points of $A$ at which $f$ is continuous in the dual continuity structure is a $G_\delta$-set residual in $A$. *S. B. Myers.*

**Braconnier, Jean. Spectres d'espaces et de groupes topologiques.** Portugaliae Math. 7, 93–111 (1948).

This is an elaboration and expansion of the usual set of theorems and proofs about limit spaces and limit groups, both direct and inverse. The original definition is so formulated that one specialization gives direct limits, another gives inverse limits. Most of the theorems deal with properties of subspaces, subgroups, continuous mappings, and homomorphisms, which are preserved under passing to the limit, and the results are not very different from previous results in this direction. The terminology is exclusively that of Bourbaki. *W. Ambrose* (Cambridge, Mass.).

**Alexiewicz, A. Linear functionals on Denjoy-integrable functions.** Colloquium Math. 1, 289–293 (1948).

Two topologies are considered on the space $(D)$ of Denjoy integrable functions over an interval $[a, b]$. The first is a norm topology in which $\|x\| = \max_{a \leq s \leq b} |(D)\int_a^s x(t)dt|$. The second is a sequential topology in which $\lim_n x_n = x_0$ if the sequence $(D)\int_a^s x_n(t)dt$ is (1) uniformly bounded, (2) convergent in measure to $(D)\int_a^s x_0(t)dt$, (3) convergent to $(D)\int_a^b x(t)dt$ if $s=b$. In each case the general linear functional over the space has the form $F(x) = (D)\int_a^b x(t)h(t)dt$, where in the first topology $h(t)$ is of bounded variation, $h(b)=0$ and $\|F\| = \text{var}_a^b h(t)$; and in the second $h(t)$ is absolutely continuous. *R. E. Fullerton* (Madison, Wis.).

**Mikusiński, J. G.-, and Ryll-Nardzewski, C. On linear functionals in Abelian groups.** Colloquium Math. 1, 294–296 (1948).

Let $E$ be an additive Abelian group with a Fréchet sequential topology. A subset $G$ of $E$ is said to be totally dense if for any $x \varepsilon E$ and any sequence $\{x_n\}$ for which $\lim_n x_n = x$, $x_n \neq x$, there exists a double sequence $\{x_{nk}\} \subset G$ such that (1) $\lim_k x_{nk} = x_n$, (2) for every sequence $l_i$ of positive integers there exist sequences $\{n_i\}$ and $\{k_i\}$ with $n_{i+1} > n_i$, $k_i > l_{n_i}$ and $\lim_i x_{n_i k_i} = x$. It is shown that an additive continuous functional on a totally dense subgroup of $E$ can be uniquely extended to an additive continuous functional on all of $E$. From this result the authors prove the second representation theorem of Alexiewicz above.

*R. E. Fullerton* (Madison, Wis.).

**Kantorovič, L. V. On the general theory of the approximate methods of analysis.** Doklady Akad. Nauk SSSR (N.S.) 60, 957–960 (1948). (Russian)

This paper formulates a general process for estimating the errors in the approximate solution of certain equations involving linear operators. The arrangement is such that the original and approximating equations can relate to different pairs of vector spaces. Thus let $U$, $V$, $\bar{U}$, $\bar{V}$ be normed linear spaces, and let $K$, $\bar{K}$ be linear operators on $U$ to $V$ and on $\bar{U}$ to $\bar{V}$, respectively. We consider the pair of equations (i) $Ku = v$ and (ii) $\bar{K}\bar{u} = \bar{v}$. Suppose that $\phi$ is an isomorphism from a subspace $U'$ of $U$ onto $\bar{U}$, and $\psi$ an isomorphism from a subspace $V'$ of $V$ onto $\bar{V}$; and that the following conditions are fulfilled: (I) $\|\bar{K}\phi u' - \psi K u'\| < \epsilon \|u'\|$ for each $u' \varepsilon U'$; (II) for each $u \varepsilon U$ there exists a $u' \varepsilon U'$ such that $\|\psi K u - \psi K u'\| < \epsilon_1 \|Ku\|$ and $\|u'\| < M\|Ku\|$. Then the author states, without proof, theorems as follows. (1) If $K^{-1}$ exists ($\|K^{-1}\| < \infty$) and $q = (\epsilon M + \epsilon_1)\|\psi^{-1}\| < 1$, then for every $\bar{v}$ equation (ii) has a solution $\bar{u}$ such that $(1-q)\|\bar{u}\| \leqq M\|\phi\| \, \|\psi^{-1}\| \, \|\bar{v}\|$. (2) If $\bar{K}^{-1}$ exists, and if it is known that there are a solution $u^*$ of (i) and a $u' \varepsilon U'$ such that $\|u^* - u'\| < \epsilon_2 \|u^*\|$, then

$$\|u^* - \phi^{-1}\bar{K}^{-1}\psi v\| \leqq \{\epsilon_2 + [\epsilon(1+\epsilon_2) + \epsilon_2 \|\psi K\|]\|\phi^{-1}\| \, \|\bar{K}^{-1}\|\} \|u^*\|.$$

(3) If we have a sequence of such approximations so that $\phi$, $\psi$, $\epsilon$, $\epsilon_1$, $\epsilon_2$, $M$, $\bar{U}$, $\bar{V}$ depend on $n$, such that $\bar{K}^{-1}$ exists and

$$\lim_{n \to \infty} M\|\phi\| \, \|\psi^{-1}\| \, \|\phi^{-1}\|[\epsilon + \epsilon_2(1 + \|\psi\|)] = 0,$$

and $\lim_{n \to \infty} \epsilon_1 \|\psi^{-1}\| = 0$ and if further $K^{-1}$ exists, then

$$\lim_{n \to \infty} [\phi^{-1}\bar{K}^{-1}\psi v - u^*] = 0.$$

(4) If (i) for each $u \varepsilon U$ there exists $u' \varepsilon U'$ such that $\|u - u'\| \leqq \epsilon_3 \|u\| + N\|Ku\|$, (ii) $\bar{K}^{-1}$ exists, and

(iii) $r = \epsilon_3(1 + \|\psi K\| \, \|\phi^{-1}\| \, \|\bar{K}^{-1}\|) + 2\epsilon \|\phi^{-1}\| \, \|\bar{K}^{-1}\| < 1$,

then $K^{-1}$ exists and

$$(1-r)\|K^{-1}\| \leqq \|\phi^{-1}\bar{K}^{-1}\psi\| + N(1 + \|\psi K\| \, \|\phi^{-1}\| \, \|\bar{K}^{-1}\|).$$

These theorems simplify somewhat in the case where $U = V$, $U' = V'$, etc., and $Ku = u - \lambda Hu$, $\bar{K}\bar{u} = \bar{u} - \lambda \bar{H}\bar{u}$. In that case, the author states an additional theorem. (5) If $H$ is completely continuous, $\bar{K}^{-1}$ exists, and a certain number $p' < 1$, then $\|K^{-1} - [I + \phi^{-1}\bar{K}^{-1}\lambda\phi H]\| < p'$. [The omission of conditions that certain $v$ be in $V'$ is in the original.] Applications to infinite systems of algebraic equations, to Fredholm integral equations, and to certain systems of ordinary differential equations, are briefly sketched.

*H. B. Curry* (State College, Pa.).

**Schmeidler, Werner. Über normalisierbare Operatoren.** Arch. Math. 1, 340–347 (1949).

Discussion of "normalizable" operators in Hilbert space, i.e., bounded linear operators which are equivalent to normal operators. It is pointed out that $A$ is normalizable if and only if $A = DL$, where $D$ is Hermitian positive definite, $D^{-1}$ bounded and $L$ satisfies $LDL^* = L^*DL$. This condition is discussed in detail in special cases (e.g., completely continuous operators) using so-called biorthogonal systems in Hilbert space, which are a generalization of orthogonal systems. *F. I. Mautner* (Cambridge, Mass.).

**Plans y Sanz de Bremond, Antonio. Linear operators in Hilbert space and their spectra.** Revista Acad. Ci. Madrid 42, 309–391 (1948). (Spanish)

This is a continuation of the author's two previous papers dealing with Hilbert space and $n$-dimensional vector spaces

[same Revista **40**, 59–100, 195–222 (1946); **41**, 197–257 (1947); these Rev. **9**, 192, 357]. The work is an extensive expository account of standard material. Among the topics considered are representation of operators by infinite matrices, the resolvent and spectrum of a closed operator, and the spectral analysis of self adjoint and normal operators.

*A. E. Taylor* (Los Angeles, Calif.).

**Plesner, A. The structure of the conjugate graph of a self-adjoint operator.** Doklady Akad. Nauk SSSR (N.S.) **66**, 557–560 (1949). (Russian)

Let $A$ be a linear transformation defined in a certain subset $\Omega$ of a unitary space $H$, satisfying $(Af, g)=(f, Ag)$ for all $f$ and $g$ in $\Omega$. The pairs $[g, g^*]$, for which the inner product $(Af, g)=(f, g^*)$ for all $f\epsilon\Omega$, form the conjugate graph $\Gamma_A{}^*$. Let $\mathfrak{P}_\lambda$ be the set of vectors $Af-\lambda f$, $\mathfrak{T}_\lambda$ the orthogonal complement of $\mathfrak{P}_\lambda$ in $H$, and $T_\lambda$ the set of pairs $[g, \tilde{\lambda}g]$, $g\epsilon\mathfrak{T}_\lambda$. It is proved that $\Gamma_A{}^*=\overline{\Gamma}_A+T_\lambda+T_{\tilde{\lambda}}$, where $\overline{\Gamma}_A$ is the closure of the graph of $A$. Also certain relations between $\overline{\Gamma}_A$, $\mathfrak{T}_\lambda$ and $\mathfrak{T}_{\tilde{\lambda}}$ are established.

*F. I. Mautner* (Cambridge, Mass.).

**Levitan, B. M. Correction to the paper "A generalization of the operation of translation and infinite hypercomplex systems."** Mat. Sbornik N.S. **24(66)**, 501–502 (1949). (Russian)

The paper appeared in the same Sbornik N.S. **17(59)**, 9–44 (1945); these Rev. **7**, 254.

**Dowker, Yael Naim. A note on the ergodic theorems.** Bull. Amer. Math. Soc. **55**, 379–383 (1949).

Examples are given of measurable transformations in a measure space for which the individual ergodic theorem holds while the mean ergodic theorem does not hold.

*N. Dunford* (New Haven, Conn.).

**Diliberto, Stephen P. Special properties of measure preserving transformations.** Bull. Amer. Math. Soc. **55**, 554–562 (1949).

The author proves the following theorems. (1) The condition that the flow defined by $dx_i/dt=P_i(x_1, \cdots, x_n)$ $(i=1, \cdots, n)$, the $P_i$ being holomorphic, preserve $p$ measure, $p$ any (fixed) integer between 1 and $n-1$, is that $\partial P_i/\partial x_j=-\partial P_j/\partial x_i$ for all $i$ and $j$. These conditions imply the motion is rigid. (2) The only point transformations, of class $C^2$, of Euclidean three-space into itself which preserve area are rigid motions (and reflections). The proofs are straightforward and self-contained. *W. H. Gottschalk.*

**Vinograd, R. On the limiting behavior of unbounded integral curves.** Doklady Akad. Nauk SSSR (N.S.) **66**, 5–8 (1949). (Russian)

The author extends earlier investigations of Poincaré [see Oeuvres, v. 1], I. Bendixson [Acta Math. **24**, 1–88 (1901)], and Solntzev [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 233–240 (1945); these Rev. **7**, 117] of a system (1) $dx/dt=P(x, y)$, $dy/dt=Q(x, y)$. Here $P$ and $Q$ are merely assumed continuous in the entire plane. The investigation deals mainly with the topology of the $\alpha$- and $\omega$-sets (in the sense of G. D. Birkhoff) of an unending trajectory $L$, written $E_\alpha(L)$, $E_\omega(L)$. Necessary and sufficient conditions for a planar set to be an $E_\omega(L)$ of some system (1) is that it be the boundary of a simply connected domain. (2) If a dynamical system (the analogue of (1)) fills up a locally compact metric space with countable base and $L$ is a trajectory which is noncompact for $t>0$, then $E_\omega(L)$ has no compact components. In particular, for Euclidean spaces this implies that if $L$ is an unbounded trajectory then $E_\omega(L)$ has no bounded components. The paper also contains further results regarding the components of the sets $E_\omega$, and of the set $E_1$ of the singular points of a given $E_\omega$. *S. Lefschetz* (Princeton, N. J.).

**Gottschalk, W. H., and Hedlund, G. A. The dynamics of transformation groups.** Trans. Amer. Math. Soc. **65**, 348–359 (1949).

Let $T$ be an Abelian transformation group acting on a space $X$, and assume that $T$ is generated by a compact neighborhood of unity. A set $S\subset T$ is called a replete semigroup if $SS\subset S$ and $S$ contains a translate of every compact subset of $T$. The usual notions of recurrence are generalized as follows. A point $x\epsilon X$ is recurrent under $T$ if $xS\bigcap U\neq 0$ for every replete $S$ and every neighborhood $U$ of $x$. The group $T$ is regionally recurrent if $U\bigcap US\neq 0$ for every replete $S$ and every nonvoid open set $U$. It is shown that if $G$ is a relatively dense closed subgroup of $T$ then $G$ is regionally recurrent if and only if $T$ is, and that a point $x$ is recurrent under $G$ if and only if it is recurrent under $T$. An example shows that a point may be nonrecurrent under every cyclic subgroup of $T$ and yet be recurrent under $T$. If $X$ is complete and metrizable and $T$ is regionally recurrent then the set $R$ of recurrent points is a dense $G_\delta$. If $X$ is separable and $\mu$ is a finite invariant Borel measure in $X$ then $\mu(X-R)=0$ (generalized Poincaré recurrence theorem). A kind of incompressibility property characterizes the case of pointwise recurrence $(R=X)$, and under certain conditions (e.g., when $X$ is locally compact and zero-dimensional) this is equivalent to pointwise almost periodicity.

*J. C. Oxtoby* (Bryn Mawr, Pa.).

**Hopf, E. Statistics of geodesic lines on manifolds of negative curvature.** Uspehi Matem. Nauk (N.S.) **4**, no. **2(30)**, 129–170 (1949). (Russian)

Translated from Ber. Verh. Sächs. Akad. Wiss. Leipzig **91**, 261–304 (1939); these Rev. **1**, 243.

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## *Calculus of Variations*

**Morse, Marston. Les progrès de l'analyse variationnelle globale et son programme.** Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 406–416 (1948).

In this address, delivered at the third National Congress of the Italian Mathematical Union, the author reviews some of his contributions to variational analysis in the large, points out its connection with bilinear and quadratic analysis, indicates recent developments in the representation and spectral theory of bilinear functionals obtained in collaboration with Transue and states three typical unsolved problems. *G. A. Hedlund* (New Haven, Conn.).

**Putnam, C. R. An application of spectral theory to a singular calculus of variations problem.** Amer. J. Math. **70**, 780–803 (1948).

Results from the spectral theory of the differential equation (1) $y''+\{\lambda-q(x)\}y=0$ are applied to the problem of minimizing the integral $J(y)=\int_0^\infty(y'^2+qy^2)dx$. When $q(x)$ is continuous and bounded below, (1) is in the limit point

case. Let $S_\alpha$ be the spectrum of the boundary value problem consisting of (1) and the condition

$$y(0)\cos\alpha+y'(0)\sin\alpha=0.$$

It is shown that $0\leqq J(y)\leqq\infty$ for all functions $y(x)$ in $L^2(0,\infty)$ which are of class $D'$ on $(0,\infty)$ and vanish at the origin, if and only if the spectrum $S_0$ contains no negative values of $\lambda$. The main tool is a lemma which generalizes the Parseval identity used in problems with a finite interval. This lemma is also applied to deduce some results on the spectrum $S_\alpha$. *J. E. Wilkins, Jr.* (Buffalo, N. Y.).

**De Franchis, Franco. Sur les trajectoires des problèmes variationnels.** Bull. Sci. Math. (2) **72**, 150–160 (1948).

The author shows that if a $(2n-1)$-parameter family of trajectories in $n$-space is characterized as the family of extremals of an "action integral" $\int L'(x, dx, E)dt$, and also as the extremals of a Hamiltonian integral $\int L(x, \dot{x})dt$, where $t$ is the time and $E$ is the kinetic energy, then we must have $dt=\partial L'(x, dx, E)/\partial E$. A converse of this has been considered by Gugino [Atti del Primo Congresso Un. Mat. Ital., 1937, pp. 360–365 (1938)]. A formal application of the result to the relativity dynamics of a particle in the field of a single large spherical mass is worked out.

*L. M. Graves* (Chicago, Ill.).

✱**John, Fritz. Extremum problems with inequalities as subsidiary conditions.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 187–204. Interscience Publishers, Inc., New York, N. Y., 1948. $5.50.

Let $F(x)$ be a real-valued function defined and of class $C'$ on the $n$-dimensional space $E_n$. Let $G(x, y)$ be a real-valued function defined for $x$ in $E_n$ and $y$ in a compact metric space $S$ such that $G$ and its derivatives with respect to the coordinates $x_i$ of $x$ are continuous on $E_n\times S$. Let $R'$ be the set of all points $x$ in $E_n$ such that $G(x, y)\geqq 0$ for all $y$ in $S$. Let $x^0$ be a point of $R'$ such that $F(x^0)$ is the minimum value of $F(x)$ for $x$ in $R'$. It is then shown that there are a finite set of points $y^1, \cdots, y^s$ $(0\leqq s\leqq n)$ in $S$ and numbers $\lambda_0, \cdots, \lambda_s$ which do not all vanish such that $G(x^0, y^r)=0$ $(r=1, \cdots, s)$, $\lambda_0\geqq 0$, $\lambda_r>0$, and such that, if

$$\phi(x)=\lambda_0F(x)-\sum_{r=1}^{s}\lambda_rG(x, y^r),$$

then $\partial\phi/\partial x_i=0$ when $x=x^0$. Conversely, if such points $y^r$ and numbers $\lambda_0$, $\lambda_r$ can be found, then $x^0$ will surely furnish $F(x)$ with a minimum value on $R'$ if in addition the matrix $\|\lambda_0\partial F(x^0)/\partial x_i \quad \partial G(x^0, y^r)/\partial x_i\|$ has rank $n$. Applications of these general results are then made to the problem of determining the sphere of minimum radius which encloses a given bounded set in $E_n$ and to the problem of determining the ellipsoid of minimum volume which encloses a given bounded set. *J. E. Wilkins, Jr.* (Buffalo, N. Y.).

**De Sloovere, H. Note sur les multiplicateurs de Lagrange dans la résolution des problèmes posés par le calcul des variations.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 735–747 (1948).

The author shows that the use of Lagrange multipliers in the problem of minimizing an $n$-tuple integral subject to differential equations interior to or on the boundary of the domain of integration leads to a system of differential equations identical with that derived in a classical method.

*J. E. Wilkins, Jr.* (Buffalo, N. Y.).

**Mangeron, D. I. Sur quelques problèmes du calcul des variations liés à la théorie d'une classe d'équations aux dérivées partielles d'ordre supérieur.** Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] **3**, 153–155 (1948).

The author announces, without proof, relations between the characteristic values of the boundary value problem $[A(x)u'+\lambda B(x)u]'+\lambda[B(x)u'+C(x)u]=0$ on $R$, $u=0$ on the boundary of $R$, and the minimum values of the functional $D(f)=\int_R A(x)f'^2(x)dx$, subject to the conditions that $H(f)=\int_R(2Bff'+Cf^2)dx=\pm 1$, and $f=0$ on the boundary of $R$. The novel aspect of the theorems lies in the interpretation of $R$ as a $2n$-dimensional rectangular domain and the symbol $'$ as designating total differentiation in the sense that $u'=\partial^{2n}u/\partial x_1\partial x_2\cdots\partial x_{2n}$. *J. E. Wilkins, Jr.*

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## *Theory of Probability*

**Varoli, Giuseppe. Probabilità.** Period. Mat. (4) **27**, 1–29 (1949).

A historical examination of the definition of probability.

*P. R. Halmos* (Chicago, Ill.).

**Petrovitch, Michel. Figures d'équilibre de deux événements ayant la même probabilité.** Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. **7**, 55–59 (1941).

Using Stirling's formula, the author derives a simple inequality for $a_{2n}=\binom{2n}{n}2^{-2n}$. He mentions the known fact that $a_{2n}$ equals the probability that during $2n$ tosses of a coin the accumulated numbers of heads and tails are never equal. *W. Feller* (Ithaca, N. Y.).

**Bolant, R. Recherche des débits admissibles dans les canalisations par le calcul des probabilités. Determination of pipes supplying and emptying numerous intermittently and independently used apparatuses.** Houille Blanche **4**, 315–330; English synopsis, 228–229 (1949).

One pipe serves several machines or installations which are used independently and at random. Elementary considerations permit one to determine the dimension of the pipe which will be sufficient during a fraction $\alpha$ of time.

*W. Feller* (Ithaca, N. Y.).

**Thionet, Pierre. Essai de détermination mathématique du stock optimum.** J. Soc. Statist. Paris **86**, 99–121 (1945).

**Dayre, J. Essai sur le foisonnement des stocks dans l'économie concurrentielle.** J. Soc. Statist. Paris **86**, 122–130 (1945).

Assume that $s$ identical shops serve $n$ customers. The authors describe a competitive and a noncompetitive system assuming, respectively, that the customers select shops at random or that each shop has $n/s$ customers as a fixed clientele. Furthermore, it is assumed that during each distribution period each customer has probability $p$ of requiring a unit amount of a certain commodity. How big should the supply of each shop be in order to reduce to $\alpha$ the probability that the supply exceeds the demand (a) in a particular shop, (b) in all shops? The authors solve this problem by elementary methods. The emphasis lies on numerical results which are collected in numerous tables. *W. Feller.*

**Baticle, Edgar. Le problème des stocks.** J. Soc. Statist. Paris **87**, 100–108; discussion, 108–109 (1946).

The author apparently intends to treat the problem of the preceding review, but compares customers to $n$ balls placed at random in $s$ compartments and assumes without explanation that all distinguishable arrangements have the same probability. [This is the Einstein-Bose statistics which attributes the same weight to the event that all customers select a particular shop as to the event that they are equidistributed.] *W. Feller* (Ithaca, N. Y.).

**Friedman, Bernard. A simple urn model.** Comm. Pure Appl. Math. **2**, 59–70 (1949).

The author considers the following problem. An urn contains $a$ white and $b$ black balls. A ball is drawn at random and then $1+\alpha$ additional balls of the same color and $\beta$ additional balls of the opposite color are added. After this procedure has been repeated $n$ times what is the probability distribution of the number of white balls? The problem is reduced to a difference-differential equation which is then solved in the three following special cases: (1) $\beta=0$, which leads to the classical Pólya-Eggenberger [G. Pólya, Ann. Inst. H. Poincaré **1**, 117–161 (1931)] distribution; (2) $\alpha=-1$, $\beta=1$, which leads to another formulation of the so called Ehrenfest model [P. and T. Ehrenfest, Phys. Z. **8**, 311–314 (1907); M. Kac, Amer. Math. Monthly **54**, 369–391 (1947); these Rev. **9**, 46]; (3) $\alpha=0$, which the author proposes as a simple model for a safety campaign and which leads to the Poisson distribution. *M. Kac.*

**Woodbury, Max A. On a probability distribution.** Ann. Math. Statistics **20**, 311–313 (1949).

Let $P(n, x)$ be the probability of $x$ successes in $n$ trials, $p_x$ the probability of the event after $x$ previous successes, and $q_x=1-p_x$. The author shows that

$$P(n, x)=p_0p_1\cdots p_{x-1}F(n, x),$$

where

$$F(n, x)=\sum_{i=0}^{x}\{q_i^x/[(q_i-q_0)\cdots(q_i-q_{i-1})(q_i-q_{i+1})\cdots(q_i-q_x)]\}.$$

An analogue of the Poisson distribution is obtained by setting $p_x=\lambda_x/n$ if $n\to\infty$. *L. A. Aroian.*

**Dilworth, R. P. Note on the strong law of large numbers.** Amer. Math. Monthly **56**, 249–250 (1949).

In the Bernoulli case (1) an inequality is established, (2) a known one deduced and applied to prove the strong law of large numbers. The author states that "it is well known that the Tchebycheff inequality is not sharp enough for the strong law." He is undoubtedly referring to a direct application of the inequality; however, it should be pointed out that Tchebycheff's inequality shows that, as $n\to\infty$, $f_{\varphi(n)}-p\to 0$ almost-certainly, where $\varphi(n)=n^2$, and, as for $0\leqq\nu\leqq 2n$ we have $|f_{\varphi(n)+\nu}-f_{\varphi(n)}|\leqq 2/n$, it follows that $f_n-p\to 0$ almost-certainly. *M. Loève* (Berkeley, Calif.).

**Gnedenko, B. V. On a local theorem for the region of normal attraction of stable laws.** Doklady Akad. Nauk SSSR (N.S.) **66**, 325–326 (1949). (Russian)

Let $\xi_1, \xi_2, \cdots$ be mutually independent random variables with a common distribution. This common distribution is supposed concentrated at a set of points of the form $a+kh$, $k=0, \pm 1, \cdots$, where $h>0$ and $h$ is called maximal if it is the maximum number for which this type of representation is possible. Define $P_n(k)=\Pr\{\sum_1^n\xi_j=na+kh\}$. The following theorem is stated:

$$\frac{bn^{1/\alpha}}{h}P_n(k)-p\left(\frac{kh-A_n+na}{bn^{1/\alpha}}\right)\to 0$$

uniformly in $k$, where $p(x)$ is the density of a stable law with exponent $\alpha$, if and only if the distribution of $(\sum_1^n\xi_j-A_n)/bn^{1/\alpha}$ is asymptotically stable with density $p(x)$ and $h$ is maximal. This theorem was proved in the Gaussian case ($\alpha=2$) in an earlier paper [Uspehi Matem. Nauk (N.S.) **3**, no. 3(25), 187–194 (1948); these Rev. **10**, 132]. *J. L. Doob.*

**Kolmogorov, A. N. The solution of a problem in the theory of probability, connected with the question of the mechanism of the formation of strata.** Doklady Akad. Nauk SSSR (N.S.) **65**, 793–796 (1949). (Russian)

It is assumed that strata are formed by periods of sedimentation followed by erosion. If $h_n$ is the height of a stratum at the end of the $n$th period, $\delta_n=h_n-h_{n-1}$, $n=1, 2, \cdots$, are assumed to be mutually independent random variables with a common density function $g(x)$, and a positive expectation. Let $\zeta_n^{(\nu)}=\delta_n+\cdots+\delta_{n+\nu}$. By the strong law of large numbers, $\lim_{\nu\to\infty}\zeta_n^{(\nu)}=+\infty$ and $\varphi_n=\min_\nu\{\zeta_n^{(\nu)}\}$ is therefore determinate. If $\varphi_n\leqq 0$, the deposit from the $n$th period finally disappears, and if $\varphi_n>0$, the residue approaches $\varphi_n$. Let $f(x)$ be the density of distribution of $\varphi_n$ and let $p=\Pr\{\varphi_n>0\}=\int_0^\infty f(x)dx$. It is shown that $f(x)$ is the unique solution of the integral equation

$$f(x)=pg(x)+\int_{-\infty}^{0}g(x-y)f(y)dy,$$

and that it can be obtained by a simple iterative procedure. The conditional distribution of the final depth of a given stratum, assuming that it is positive, then has density $f(x)/p$, for $x>0$. *J. L. Doob* (Ithaca, N. Y.).

**Tortrat, Albert. Sur les fonctions de corrélation des processus de Markoff.** C. R. Acad. Sci. Paris **228**, 1559–1561 (1949).

The author considers a continuous parameter Markov process. This defines a continuous family $T(s, t)$ of operators on a Banach space [cf. Yosida and Kakutani, Ann. of Math. (2) **42**, 188–228 (1941); these Rev. **2**, 230] satisfying $T(s, t)=T(s, u)T(u, t)$ if $s<u<t$. If $T$ is continuous in $s, t$ (uniform operator topology) $T$ has the form $\alpha(s)\cdot\alpha^{-1}(t)$. In the stationary case $T(s, t)$ becomes $T(t-s)$ and (same continuity hypothesis) $T(t)$ can be written in the form $Je^{ta}$. The spectral properties of the operator $a$ are investigated and related to those of the covariance function of the process. *J. L. Doob* (Ithaca, N. Y.).

**Loève, Michel. Indicateurs abstraits et champs stochastiques.** C. R. Acad. Sci. Paris **228**, 1561–1564 (1949).

An abstract formulation of random variable and expectation is outlined, general enough to include, for example, the following: random variable is a self-adjoint Hilbert space operator obtained as the integration of a function with respect to a given resolution of the identity; the expectation of the operator $X$ is the trace of $PX$ where $P$ is an operator of the same type which is nonnegative and has trace 1. *J. L. Doob* (Ithaca, N. Y.).

**Bochner, S. Diffusion equation and stochastic processes.** Proc. Nat. Acad. Sci. U. S. A. **35**, 368–370 (1949).

The author considers Markov processes whose transition probability density functions $f(s, x, t, y)$ (the probability

of a transition from $x$ at time $s$ to $y$ at time $t$) satisfy (1) $\partial f/\partial s=-A_s(x)f$, $\partial f/\partial t=A_t(y)f$. The operators $A_s(x)$ and $A_t(y)$ are usually written explicitly as integro-differential operators, but it is stated that the hypothesis $f\to\delta(x, y)$ (Dirac function) when $t-s\to 0$ always implies these equations with $A_s(x)$ a linear operator for each $s$ acting on functions of $x$ and $A_s^*(x)$ its adjoint. In particular, in the stationary case the operator $A_s$ becomes $A$, independent of $s$, and it is stated that if $A$ is self-adjoint and if $e^{-u\Phi(\lambda)}$ is completely monotone in $\lambda$ for each $u$, replacing $A$ by $\Phi(A)$ yields a new stochastic process whose transition probability density is written down explicitly. For example, if $\Phi(\lambda)=\lambda^p$, $0<p<1$, and if $A$ is a second order elliptic differential operator, the new process is a process with independent increments, whose increments have stable distributions. In the stationary case $f(s, x, t, y)=f(t-s, x, y)$ and the first equation of (1) becomes, after the operation just described, $\partial f(u, x, y)/\partial u=A^p f$. The equation $\partial f/\partial u=\{(A+c)^p-c^p\}f$ also defines a process. The above procedure of replacing $A$ by $\Phi(A)$ is generalized, to obtain nonstationary transition probabilities. One of the principal difficulties is, as usual, to show that the solution of (1) defines a stochastic process, in particular, that the solution is nonnegative and integrates to one in the variable describing the final position. The present paper is sketchy, and a full treatment is to appear later. *J. L. Doob* (Ithaca, N. Y.).

**Yosida, Kôsaku. Brownian motion on the surface of the 3-sphere.** Ann. Math. Statistics **20**, 292–296 (1949).

Let $S$ be an $n$-dimensional compact Riemann space such that the totality $G$ of the isometric transformations of $S$ onto $S$ constitutes a Lie group transitive on $S$. A Brownian motion on $S$ is defined as a Markov process with stationary transition probabilities which are invariant under a transformation of the group $G$. It is proved that there is one and only one Brownian motion on the surface of a three-dimensional sphere, and the corresponding transition probabilities are calculated explicitly. *J. L. Doob.*

**Ammeter, Hans. Die Elemente der kollektiven Risikotheorie von festen und zufallsartig schwankenden Grundwahrscheinlichkeiten.** Mitt. Verein. Schweiz. Versich.-Math. **49**, 35–95 (1949).

In a previous paper [Skand. Aktuarietidskr. **31**, 171–198 (1948); these Rev. **10**, 463], the author studied some of the modifications to be made in the collective risk theory when the basic probabilities vary in a prescribed manner or according to chance. In the present paper the author studies the detailed application of his theory to many special cases. Comparisons between empirical distribution functions and distribution functions obtained on the basis of the author's theory are made. *J. Wolfowitz.*

***Pestalozzi, Anton. Untersuchungen über den Verlauf des latenten Fehlbetrages bei offenen Versicherungskassen.** Thesis, Eidgenössische Technische Hochschule, Zürich, 1948. 79 pp.

The author studies the development of an institution for social insurance and makes the following assumptions. (1) All members, charter members as well as members entering at a later date, pay the same average premium. (2) The membership at any time $t$ is given by a function $H(t)$. (3) The age distribution of the charter members is given by a function $H(0, x)$, where $H(0, x)$ is the number of persons of age $x$ entering at time $t=0$. (4) All persons entering at a later date are of the same age $z$ at the time of their admittance. If the actual membership of such a group is observed it is seen that no financial equilibrium exists at any time so that a "latent deficit" appears. This is a consequence of assumption (1) which compels future generations to pay part of the cost of the insurance of the initial group. Financial equilibrium exists only for the body consisting of both the initial group and the future entries. Let $P$ be the average premium paid by all members and $P_z$ the individual premium corresponding to the age at entry $z$. The latent deficit is then given by

$$L(t)=(P-P_z)a_z\int_t^\infty v^{k-t}E(k)dk$$

and the mean latent deficit by $l(t)=L(t)/H(t)$. In these formulae $E(k)$ is the renewal function of the aggregate of insured persons, $v$ is the rate of discount and $a_z$ the annuity for a person of age $z$. The author assumes that the function $H(t)$ is known and discusses the renewal function and the latent deficit. H. Richter [Math. Ann. **118**, 145–194 (1941); these Rev. **6**, 11] gave conditions for the stability and the stability in the mean of the renewal function. The author obtains corresponding theorems for the latent deficit.

*E. Lukacs* (China Lake, Calif.).

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## *Mathematical Statistics*

**Schneider, Stanislas. Sur l'ajustement des courbes à branches limitées.** J. Soc. Statist. Paris **89**, 218–227 (1948).

The author is interested in the graduation of observed distributions with finite range which are not well graduated in the tails by the normal probability function. He suggests the use of the probability function (*) [see below], essentially due to Laplace, of sample sums if $x_1, x_2, \cdots, x_n$, are drawn from a rectangular population, or from a series of rectangular populations with variable ranges. More precisely let $z=x_1+x_2+\cdots+x_n$, where $p(x)=1/(2a)$, $-a\leqq x\leqq a$; then (*) is given by

$$p(z)=\{1/(2a)^n(n-1)!\}\sum_{i=0}^{k}(-1)^i{}_nC_i[z+(n-2i)a]^{n-1},$$

$(2k-n)a\leqq z\leqq[2(k+1)-n]a$, $k$ being the greatest integer for which $z+(n-2i)a>0$. In the more general case with varying $a$, $a_i$, (*) is too complicated to be given here. The method of moments is given and two tables of (*) are provided. A numerical example illustrates the theory.

*L. A. Aroian* (New York, N. Y.).

**Selberg, Henrik L. On the corrections for grouping.** Skand. Aktuarietidskr. **30**, 179–190 (1947).

Let $F(x)$ be a continuous distribution function,

$$\mu_r=\int_{-\infty}^{\infty}x^r dF(x)$$

and $\mu_r(\xi, \omega)$ the uncorrected moments,

$$\mu_r(\xi, \omega)=\sum_{\nu=-\infty}^{\infty}(\xi+\nu\omega)^r\{F(x+\omega/2)-F(x-\omega/2)\},$$

obtained by grouping into intervals of length $\omega$ with middle-points $\xi+\nu\omega$, $\nu=0, \pm1, \pm2, \cdots$; then Sheppard's formulas for the approximate value of $\mu_r$ are

$$\mu_r\sim\sum_{\nu=0}^{r}\omega^\nu\binom{r}{\nu}(2^{1-\nu}-1)B_\nu\mu_{r-\nu}(\xi, \omega),$$

where $B_\nu$ are the Bernoulli numbers defined by

$$t(e^t-1)^{-1}=\sum_{\nu=0}^{\infty}B_\nu t^\nu/\nu!.$$

Further let $\eta_r(\xi,\omega)$ represent the difference between the exact value of $\mu_r$ and the approximate value of $\mu_r$ given by Sheppard's formula. The author shows that

$$\eta_r(\xi,\omega)=i^{-r}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{d^r}{dt^r}\left\{\frac{\omega t}{\omega t-2n\pi}\varphi(t-2n\pi/\omega)e^{2in\pi t/\omega}+\frac{\omega t}{\omega t+2n\pi}\varphi(t+2n\pi/\omega)e^{-2in\pi t/\omega}\right\}_{t=0},$$

where $\varphi(t)=\int_{-\infty}^{\infty}e^{itx}dF(x)$, under the assumption of the existence of the absolute moment $\int_{-\infty}^{\infty}|x|^r dF(x)$. An application is made to the distribution of sample sums.

*L. A. Aroian* (New York, N. Y.).

**Risser, R. Essai sur les courbes de distribution statistique.** J. Soc. Statist. Paris **89**, 288–306 (1948).

Also published under the same title in Assoc. Actuair. Belges. Bull. no. **54**, 41–72 (1948) [these Rev. **10**, 386]. The present paper lacks the complementary note.

*L. A. Aroian* (New York, N. Y.).

**Risser, R. Note relative aux surfaces de probabilités.** J. Soc. Statist. Paris **89**, 381–409 (1948).

Also published under the same title in Assoc. Actuair. Belges. Bull. no. **53**, 5–48 (1948) [these Rev. **10**, 312].

*L. A. Aroian* (New York, N. Y.).

**Fréchet, Maurice. Sur une limitation très générale de la dispersion de la médiane.** J. Soc. Statist. Paris **81**, 67–76; discussion, 76–78 (1940).

The author compares the sample median and sample mean with respect to certain properties, principally for unimodal distributions with finite variance and for large samples. Let $X$ be a random variable with unimodal distribution, $\sigma_X$ its standard error, $\theta_X$ its mean absolute error and $E_X$ its interquartile range. Let $M$ be the median and $V$ the arithmetic mean of a sample of $n$ independent random variables distributed as $X$. Then for sufficiently large $n$, (i) $\sigma_M \leqq 1.74\sigma_X/\sqrt{n}$, (ii) $\theta_M \leqq 1.60\theta_X/\sqrt{n}$, (iii) $E_M \leqq 1.35E_X/\sqrt{n}$. For $V$, there is the equality $\sigma_V=\sigma_X/\sqrt{n}$ corresponding to and better than (i), but there are no inequalities for $V$ corresponding to (ii) and (iii). The author concludes that the sample median should be more widely used except when the distribution is known to be such that the sample mean is better.

*C. M. Stein* (Paris).

**Amy, Lucien. Étude statistique de l'expression**

$$\frac{p_1}{q_1}=\frac{p_2}{q_2}=\cdots=\frac{p_i}{q_i}=\cdots=c^{te}.$$

**Première application à l'étude de la fréquence des formules digitales.** J. Soc. Statist. Paris **81**, 39–45; discussion, 45–47 (1940).

For a study of fingerprint classifications, the author considers a statistical model consisting of mutually exclusive events $A_1, A_2, \cdots, B_1, B_2, \cdots$, the probabilities of $A_i$ and $B_i$ being $p_i$ and $q_i$, respectively, with $p_i=\rho q_i$. In a sample of $n$, with $A_i$, $B_i$ appearing $a_i$, $b_i$ times, the maximum likelihood estimate of $\rho$ is $\sum a_i/\sum b_i$. The author also considers the deviations $(a_i-b_i\rho)(\rho+1)^{-1}$ and the reduced deviations $(a_i-b_i\rho)(a_i\rho+b_i\rho)^{-\frac{1}{2}}$ which he takes as normally distributed. For fingerprints, the classes $A$ and $B$ are ages less than or greater than 22; the fingerprints themselves are drastically simplified to two types. One analysis shows poor, another good, agreement with the theory.

*J. Riordan.*

**Féron, Robert. Mérites comparés des divers indices de corrélation.** J. Soc. Statist. Paris **88**, 328–350; discussion, 350–352 (1947).

**Godwin, H. J. Some low moments of order statistics.** Ann. Math. Statistics **20**, 279–285 (1949).

The means, variances and covariances of order statistics from a normal population are obtained in terms of

$$\psi(i,j)=\int_{-\infty}^{\infty}F^i(x)\int_x^{\infty}\{1-F(y)\}^j dxdy,$$

$$\alpha(i,j)=\int_{-\infty}^{\infty}xF^i(x)\{1-F(x)\}^j dx,$$

$$\beta(i,j)=\int_{-\infty}^{\infty}x^2f(x)\{1-F(x)\}^j dx,$$

where $F(x)=\int_{-\infty}^{x}f(t)dt=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x}e^{-t^2/2}dt$. Results for sample sizes not exceeding 10 are tabulated to more significant figures than by methods used previously. An extension in the table of exact results is also given.

*H. Chernoff.*

**Quenouille, M. H. A relation between the logarithmic, Poisson, and negative binomial series.** Biometrics **5**, 162–164 (1949).

Let the probability of obtaining $n$ groups, $n=0, 1, 2, \cdots$, be given by a Poisson distribution $e^{-m}m^n/n!$, and the probability of observing $s$ individuals in any one group by $\alpha x^s/s$, $s=1, 2, \cdots$, $\alpha=-1/\log(1-x)$ (the logarithmic distribution). Then the author proves that the probability of obtaining $s$ individuals in any sample is

$$x^s(1-x)^{\alpha m}(\alpha m+s-1)!/(\alpha m-1)!s!,$$

or the $(s+1)$th term in a negative binomial series with parameter $x$ and index $\alpha m$. The result is useful in biology.

*L. A. Aroian* (New York, N. Y.).

**Madow, William G. On a source of downward bias in the analysis of variance and covariance.** Ann. Math. Statistics **19**, 351–359 (1948).

The author investigates the effects on the probabilities of errors if in the general linear univariate hypothesis the a priori assumptions concerning the means (in canonical form: that certain of the means are zero) are not satisfied. He shows that the probabilities of acceptance are increased both under the hypothesis and under the alternatives, and indicates with some numerical illustrations that substantial increases in these probabilities may result from moderate violations of the assumptions.

*E. L. Lehmann.*

**Narain, R. D. On the distribution of estimated error components in analysis of variance and covariance.** J. Indian Soc. Agric. Statistics **1**, 70–79 (1948).

Given $N$ quantities $\{Y_i\}$, subject to $m$ orthogonal linear restraints, $\sum_{i=1}^{N}c_{hi}Y_i=0$, $h=1, 2, \cdots, m$, such that $\sum_{i=1}^{N}Y_i^2$ is distributed as $\chi^2$ with $N-m$ degrees of freedom, the author derives the frequency distribution of any number (not greater than $N-m$) of the $Y_i$, say $N-p$ of them, where $p \geqq m$. This is a generalization of the results given by Irwin for $m=1$ with $c_{1i}=1$ [J. Roy. Statist. Soc. **92**, 264–267 (1929)]. The particular cases of the distribution of

$2Y$'s is presented, plus the correlation coefficient between the two $Y$'s, their respective variances, and the partial correlation between them with the other $(N-p-2)$ $Y$'s held constant. *R. L. Anderson* (Raleigh, N. C.).

**Pillai, K. C. S. A note on ordered samples.** Sankhyā 8, 375–380 (1948).

Let $x_1, x_2, \cdots, x_n$ be an ordered sample from a normal distribution. Consider the information, as defined by Fisher, of $x_i$ regarding the mean $a$ and variance $\sigma^2$. Tables are computed for $n=2, 3, \cdots, 12$; $i=1, 2, \cdots, 6$. According to these tables, the central values give the most information regarding $a$ and the extreme values give the most information regarding $\sigma^2$. An unbiased estimate of $\sigma$ which is a function of $n$ multiplied by the semi-range $\frac{1}{2}(x_n-x_1)$ is obtained and the distribution of the semi-range is studied. *H. Chernoff* (Urbana, Ill.).

**Lehmann, E. L., and Stein, C. Most powerful tests of composite hypotheses. I. Normal distributions.** Ann. Math. Statistics 19, 495–516 (1948).

In the Neyman-Pearson theory of testing composite statistical hypotheses, it is usual to consider tests associated with critical regions satisfying the condition of similarity. Thus for testing the hypothesis $H_0$ that the random variable $X=(X_1, \cdots, X_n)$ is distributed according to a probability density $f$ belonging to a certain family $F$ against the simple alternative that $X$ is distributed according to some other specified probability density $g$, it is required to find a critical region $w$ in the $n$-dimensional space of $X$ such that $\int_w g(x)dx$ becomes a maximum subject to the condition of similarity: $\int_w f(x)dx=\epsilon$ for all $f$ in $F$. It has been pointed out by Neyman, Wald, and others that it would be natural to replace the condition of similarity by the more general condition $\int_w f(x)dx \leqq \epsilon$. In the present paper, a test consisting of rejecting $H_0$ whenever $X$ takes a value in $w$ is called a most powerful test of $H_0$ against $H_1$ for level of significance $\epsilon$, if $w$ maximizes $\int_w g(x)dx$ subject to the condition $\int_w f(x)dx \leqq \epsilon$ for all $f$ in $F$. A sufficient condition for a most powerful test is deduced by means of an adaptation of the well-known fundamental lemma due to Neyman and Pearson. This condition is applied to the testing of various hypotheses concerning normally distributed variables against simple alternatives. Thus, e.g., when $X_1, \cdots, X_n$ are independent and normally distributed, and $H_0$ is Student's hypothesis that each $X_i$ has the mean 0 and an unknown variance $\sigma^2$ independent of $i$, while under $H_1$ each $X_i$ has the mean $\xi_1$ and the variance $\sigma_1^2$, it is shown that, for $\epsilon<\frac{1}{2}$, the most powerful test of $H_0$ against $H_1$ is associated with a nonsimilar region of the form $\sum(X_i-\eta)^2 \leqq k$, where $\eta$ and $k$ depend on $\xi_1$, $\sigma_1$ and $\epsilon$. Further, the concept of admissibility introduced by Wald is generalized to the present conditions. *H. Cramér* (Stockholm).

**Lehmann, E. L., and Stein, C. On the theory of some non-parametric hypotheses.** Ann. Math. Statistics 20, 28–45 (1949).

The authors consider what are called hypotheses of invariance, an example of which is the hypothesis that the joint distribution of $x_1, \cdots, x_n$ is invariant under all permutations of the variables. Most powerful tests of hypotheses of invariance against a single alternative are obtained; under certain conditions, most stringent tests of hypotheses of invariance against a class of alternatives are obtained. For example, if $Z_{ij}$, $j=1, \cdots, s_i$, $i=1, \cdots, n$, are chance variables, denote by $H$ the hypothesis that for each $i$ the joint distribution is invariant under permutations of the $Z_{ij}$, and by $H_1$ the alternative that the $Z_{ij}$ are independent, normal, with the same variance, and $E(Z_{ij})=ax_{ij}+b_i$, where $a$, $x_{ij}$, $b_i$ are given, and $a>0$, $\sum x_{ij}=0$. The most powerful test of $H$ against $H_1$ is that based on the statistic $W=\sum x_{ij}Z_{ij}$. The most stringent test of $H$ against the class of $H_1$'s, $-\infty<a<\infty$, is that based on $|W|$. Special cases of $W$ have been proposed by Pitman and Fisher. A more general definition of hypothesis of invariance is proposed, and again the most powerful test against a single alternative is described. *D. Blackwell* (Washington, D. C.).

**Berger, Agnes, and Wald, Abraham. On distinct hypotheses.** Ann. Math. Statistics 20, 104–109 (1949).

Two composite hypotheses $H_0$, $H_1$ are said to be distinct if there is a region $W$ of sample space such that for no $h_0 \epsilon H_0$, $h_1 \epsilon H_1$ is $P(W|h_0)=P(W|h_1)$. Conditions under which there exists a test of $H_0$ against $H_1$ whose maximum probability of error is less than $\frac{1}{2}$ are given. Existence of such a test implies distinctness of $H_0$ and $H_1$. *D. Blackwell.*

**Chernoff, Herman. Asymptotic Studentization in testing of hypotheses.** Ann. Math. Statistics 20, 268–278 (1949).

Methods are developed for obtaining "asymptotically similar" tests, that is, tests the size of which is constant except for an error term of prescribed order in the number of observations. The procedures are based on a statistic $t$, which would provide a good test $t \leqq c(\theta)$ of given size if the nuisance parameter $\theta$ were known, and an estimate $\hat{\theta}$ of $\theta$. The test $t \leqq c(\hat{\theta})$ is taken as a first approximation, and further approximations are obtained by using successively as corrections for the boundary of $t$ the previous error term evaluated at $\theta=\hat{\theta}$. The properties of this procedure and of certain modifications of it are discussed under various regularity assumptions. As an illustration the method is applied to a problem treated previously by Welch [same Ann. 18, 118–122 (1947); these Rev. 8, 474] from a different point of view. *E. L. Lehmann* (Berkeley, Calif.).

**Tukey, John W. Sufficiency, truncation and selection.** Ann. Math. Statistics 20, 309–311 (1949).

"If a family of distributions admits a set of sufficient statistics, then the family obtained by truncation to a fixed set, or by fixed selection, also admits the same set of sufficient statistics." *J. Wolfowitz* (New York, N. Y.).

**Pitt, H. R. On the theory of statistical procedures.** Proc. Cambridge Philos. Soc. 45, 354–359 (1949).

Remarks on the theory of statistical decision functions. The author is apparently unaware of the results published by Wald after 1939. *J. Wolfowitz* (New York, N. Y.).

**Owen, A. R. G. Ancillary statistics and fiducial distributions.** Sankhyā 9, 1–18 (1948).

From the author's summary. The problem of the simplest type in insufficient estimation is considered; namely, that in which information about a single parameter is contained in a pair of statistics, which jointly constitute an exhaustive set. It is shown that if an ancillary statistic exists, the fiducial distribution defined by its aid is unique, and that the ancillary may be expressed as a function of two functionally independent pivotal quantities. . . . Examples are given to illustrate various situations which may be encountered. Two methods are discussed by which the process of fiducial inference may be exhibited geometrically. The rela-

tion of ancillary statistics to the minimal variance properties of estimators in conditional inference is considered. The Nile problem is posed as a question in the theory of differential equations. *H. Cramér* (Stockholm).

**Steinhaus, H. Elementary inequalities between the expected values of current estimates of variance.** Colloquium Math. **1**, 312–321 (1948).
Expository paper. *J. Wolfowitz* (New York, N. Y.).

**Howell, John M. Control chart for largest and smallest values.** Ann. Math. Statistics **20**, 305–309 (1949).

**Birnbaum, Z. W., and Zuckerman, H. S. A graphical determination of sample size for Wilks' tolerance limits.** Ann. Math. Statistics **20**, 313–316 (1949).

**Grubbs, Frank E. On designing single sampling inspection plans.** Ann. Math. Statistics **20**, 242–256 (1949).

In designing single sampling inspection plans, a problem is to find the acceptance number, $c$, and the smallest sample size, $n$, such that if the fraction defective of the material inspected is equal to an acceptable value, $p_1$, a large percentage of such lots will be accepted, whereas if the fraction defective is objectionable and equal to $p_2$ (where $p_1<p_2$), then a large percentage of such lots will be rejected. A solution to this problem for the case where the lot size is large compared to the sample size is given and tables are provided for quick determination of the sample size $n$ and acceptance number $c$. *From the author's summary.*

**Champernowne, D. G. Sampling theory applied to autoregressive sequences.** J. Roy. Statist. Soc. Ser. B. **10**, 204–231; discussion, 231–242 (1948).

The author studies estimation problems for stationary processes satisfying difference equations of the form

$$\sum_{i=0}^{m} b_i(X_{t-i}-c)=\epsilon_t,$$

where the $\epsilon_t$'s are mutually independent and normally distributed with zero means and variance $\alpha^2$. Estimates are found for $c$, the $b_i$'s and $\alpha^2$ in terms of $X_j$ samples, and also for the regression coefficient $\lambda$ if $z_t=\lambda Y_t+X_t$, where $X_t$ is as above. The methods involve a mixture of maximum likelihood, least squares, and Bayes's rule. [For an attack on the same type of problem, from a different point of view, see Mann and Wald, Econometrica **11**, 173–220 (1943); these Rev. **5**, 129.] *J. L. Doob* (Ithaca, N. Y.).

**Quenouille, M. H. Approximate tests of correlation in time-series.** Proc. Cambridge Philos. Soc. **45**, 483–484 (1949).

Let $x_t$ be a time-series with first serial coefficient $r_1$. Using the null hypothesis that $x_t$ is a Markov process $x_t=\rho x_{t-1}+\epsilon_t$, the author gives an approximate method for assigning confidence limits for $\rho$ of the type $r_1-c$, $r_1+c$, and shows that the limits and their probability asymmetry may be evaluated by the use of Student's distribution.
*H. Wold* (Uppsala).

**Bartlett, M. S. Corrigenda to the paper "On the theoretical specification and sampling properties of autocorrelated time-series."** J. Roy. Statist. Soc. Ser. B. **10**, no. 1, 1 p. (1948).
The paper appeared in Suppl. J. Roy. Statist. Soc. **8**, 27–41 (1946); these Rev. **8**, 283.

**Anscombe, F. J. The validity of comparative experiments.** J. Roy. Statist. Soc. Ser. A. **111**, 181–200; discussion, 200–211 (1948).

The paper and discussions following it revolve around the difficulties encountered in interpreting data from experiments which cannot be repeated under identical conditions an arbitrary number of times and also in interpreting data from one single stochastic process. One then has to agree on the use of a mathematical model which regards the individual items as drawn from different populations and the hypotheses tested can concern only relations between these distributions, as for instance relations between the distribution functions of two sequences, or between one distribution function and the following one. In addition to a mathematical model supposedly describing the phenomenon one also has to agree on a system of principles of statistical inference of which the author discusses three.

The first is Jeffreys' theory of probability in which inferences are made by application of the Bayes theorem. The second is the Neyman-Pearson theory and the third is R. A. Fisher's theory of fiducial probability. The author draws attention to the fact that none of these principles has convincing force, but has first to be accepted on grounds of expediency before inferences can be made. The author distinguishes between experiments in scientific research for which he finds the Neyman-Pearson theory most satisfactory, and experiments that lead to administrative decisions for which he prefers Jeffreys' theory of probability. Wald's theory, in which decisions are determined that minimize the maximum risk, which, to the reviewer, seems most appropriate at least for making administrative decisions, is not considered by the author or any of the discussants.
*H. B. Mann* (Berkeley, Calif.).

**Kempthorne, O., and Federer, W. T. The general theory of prime-power lattice designs. II. Designs for $p^n$ varieties in blocks of $p^s$ plots, and in squares.** Biometrics **4**, 109–121 (1948).

This is a continuation of an earlier paper [Kempthorne, Biometrika **34**, 255–272 (1947); these Rev. **10**, 313] and together with part I [same vol., 54–79 (1948); these Rev. **10**, 52] provides a thorough treatment of the designs in the title. The present part discusses the following designs: designs for $p^n$ varieties in $p^s$ plots, designs where different block sizes are used in different replicates, and designs which use the orthogonal properties of sets of orthogonal Latin squares. The latter group comprises the lattice squares but the authors propose to extend the name lattice square to the whole group. In all these designs the varietal effects are considered as resulting from the combination of $n$ factors at $p$ levels each so that the theory of factorial designs may be applied. The main feature of these designs then is that lower order interactions and main effects are partially confounded and higher order interactions preserved as far as possible. The authors indicate the application of these principles to the analysis and construction of the designs. They also give formulae for the average efficiencies and provide a

considerable number of examples for the reader who wants to familiarize himself with these designs.

*H. B. Mann* (Berkeley, Calif.).

**Federer, Walter T. The general theory of prime-power lattice designs. III. The analysis for $p^3$ varieties in blocks of $p$ plots with more than 3 replicates.** Biometrics 5, 144–161 (1949).

In continuation of previous papers [see the preceding review] the author gives a numerical example illustrating the analysis of a $3^3$ lattice design with 4 replicates. The extension to $p^n$ designs with more than $n$ replicates when $p$ is a prime is immediate. *H. B. Mann.*

**Banerjee, K. S. A note on weighing design.** Ann. Math. Statistics 20, 300–304 (1949).

The author gives a detailed discussion of weighing designs derived from factorial designs with fractional $(2^s-1)/2^s$ replications. Designs of this type were first proposed by Kempthorne [same Ann. 19, 238–245 (1948); these Rev. 9, 603]. The author then considers the optimum weighing designs proposed by Mood [same Ann. 17, 432–446 (1946); these Rev. 8, 478], and shows that these designs will yield independent estimates if one weighing is made to determine the bias in the scales. The matrix of the resulting regression equations is obtained by considering the bias as if it were an object weighed at every weighing. Thus the matrices $L_n$ of Mood's designs must be bordered by a row 10 ... 0 and a column consisting of ones. The estimates of the weights themselves are then independent of each other but not of the estimate of the bias. *H. B. Mann.*

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## *Mathematical Economics*

**Patinkin, Don. The indeterminacy of absolute prices in classical economic theory.** Econometrica 17, 1–27 (1949).

The author's main thesis is that there exists a fundamental contradiction in classical economic theory. The inconsistency arises out of the assumption that the demand for goods depends only on relative prices, whereas behavior in the monetary sector depends on absolute prices. The economic model employed is essentially the macroeconomic exchange economy analysis of Cassel. Let there be $n$ commodities, the $n$th commodity being paper money, and $p_i$ the price of the $i$th good. Then

$$D_i = f_i(p_1, p_2, \cdots, p_{n-1}), \tag{1}$$

$$S_i = g_i(p_1, p_2, \cdots, p_{n-1}), \tag{2}$$

$$D_i = S_i \tag{3}$$

($i=1, \cdots, n$). These are the demand functions, supply functions, and equilibrium conditions, respectively. By virtue of interdependence there are $n-1$ independent equations in (3); hence there are $3n-1$ equations in the $3n-1$ variables.

The general proposition on which the analysis is based is stated in the following theorem. If every equation of a system of $K$ independent equations in $K$ variables is homogeneous of some degree $t$ in the same set of variables, then the system possesses no solution, with the possible exception of the zero solution [see the following review]. The classical assumptions involved in the Casselian model is that the values of $D_i$ and $S_i$ ($i=1, \cdots, n-1$) depend only on relative prices. Hence the functions (1) and (2) are homogeneous of zero degree, and depend on the $n-2$ ratios $p_1/p_{n-1}, \cdots, p_{n-2}/p_{n-1}$. It is further assumed that the equation dropped from (3) is one referring to commodities. Thus there are $3n-4$ equations determining the $3n-4$ variables $D_i$, $S_i$, and $p_j/p_{n-1}$ ($j=1, \cdots, n-2$). The monetary equation may be written as (4) $M-p_{n-1}K\sum_{i=1}^{n-1}(p_i/p_{n-1})S_i=0$. It is now held that, on the basis of the fact that if the $g_i$ ($f_i$) ($i=1, \cdots, n-1$) are homogeneous of the zero degree in all the variables then $f_n$ ($g_n$) is homogeneous of the first degree in the same variables, and of the general proposition, the Casselian system is inconsistent, i.e., overdetermined. It follows also that since (4) cannot be homogeneous of degree one unless $K$ and $M$ are not constant, the monetary equation is inconsistent with the homogeneity assumption.

It is similarly held that various modifications are also inconsistent. The system considered by Modigliani [Econometrica 12, 45–88 (1944)] in which at least one price is absolute, is overdetermined. The system of Lange, in which there is assumed one marketable asset and where the excess demand functions are assumed homogeneous of degree zero in all prices except the rate of interest, is overdetermined. Finally, the assumption of Say's law, since it reduces the system to $n-2$ equations in $n-1$ variables, is underdetermined. A system of equations is suggested which avoids these difficulties. Every excess demand function is made to depend on the value of all assets in the economy, i.e., the absolute price level appears everywhere in the system.

*M. P. Stoltz* (Providence, R. I.).

**Frisch, Ragnar. On the zeros of homogeneous functions.** Econometrica 17, 28–29 (1949).

It is noted that the original formulation of the general proposition stated in the preceding review was deficient in that it did not admit the possibility of a zero solution. The conditions under which there exists a well-defined zero solution for different values of the constant $t$, where $t$ has the usual meaning in $f(\lambda x)=\lambda^t f(x)$, are considered.

*M. P. Stoltz* (Providence, R. I.).

**Castoldi, Luigi. Sopra un problema di massimo in meccanica economica.** Atti Accad. Ligure 4 (1947), 65–74 (1948).

Write $p_s$ for the price of the $s$th consumption good, and $x_{rs}$ for the rate of its consumption by the $r$th individual. If the total available amount $\sum_r x_{rs}$ is given for each $s$, and if the individual utility function $\Phi_r(x_{r1}, x_{r2}, \cdots)$ and the income $R_r=\sum_s p_s x_{rs}$ are given for each $r$, then the maximizing of every $\Phi_r$ is said to determine the "equilibrium" values of the $x$'s and $p$'s, with the following properties: (1) for every $r$ and $s$, $\Phi_{rs}/\Phi_{r1}=p_s/p_1$, where $\Phi_{rs}\equiv\partial\Phi_r/\partial x_{rs}$; (2) a displacement of the $x$'s and $p$'s from their equilibrium values diminishes at least one $\Phi_r$ (i.e., inflicts damage upon at least one consumer).

According to the author, Pareto regarded these conditions as maximizing not only every individual $\Phi_r(x_{r1}, x_{r2}, \cdots)$ but also some properly defined "collective utility function" of the $x$'s and $p$'s; most later writers (Amoroso, Demaria) rejected such an interpretation. The author revives Pareto's approach by considering the case when the government fixes the incomes $R_r$ so as to maximize a linear combination $\Phi=\sum\Phi_r/c_r$. Then, in addition to (1), (2) he obtains (3) for every $r$ and $s$, $\Phi_{rs}/\Phi_{r1}=c_r/c_1$. The weights $1/c_r$ are nonnegative. The maximum value for $\Phi$ is smallest when the weights are all equal; it is largest when only one is positive. Comparability of individual utilities seems to be assumed throughout, in the sense that every utility function is deter-

mined up to the same arbitrary transformation, independent of $r$. *J. Marschak* (Chicago, Ill.).

**Roy, René. Théorie des choix.** J. Soc. Statist. Paris **85**, 134–138 (1944).

This note is in answer to objections raised against certain properties of a price index defined in an earlier memoir [De l'utilité, contribution à la théorie des choix, Actual. Sci. Ind., no. 930, Hermann, Paris, 1942; these Rev. **8**, 217]. The properties in question are (1) $W=\omega/z$, (2) $z^{-1}dz=\sum qdp/\sum qp$, where $W$ is the marginal utility of the money income, $\omega$ the marginal utility of the real income, $z$ the price index, and $p$ and $q$ the prices and quantities respectively. The calculation details show, under certain utility equivalence assumptions, that the following relations hold:

$$(1') \qquad W=(1+\epsilon)\omega/z,$$

$$(2') \qquad z^{-1}dz=(1+\epsilon)\sum qdp/\sum qp-\epsilon dr/r,$$

where $\epsilon$ is a function which is identically zero only for particular forms of the utility function, but which in any case is considered negligible. The condition that $z$ does not depend on $r$ (the nominal income) can be expressed by either of two conditions $\sum q\partial U/\partial q=$constant or $\sum p\partial U/\partial p=$constant, where $U$ is the utility function. *M. P. Stoltz.*

## TOPOLOGY

**Eilenberg, Samuel. On the problems of topology.** Ann. of Math. (2) **50**, 247–260 (1949).

Revue des principaux problèmes non résolus dans le domaine de la topologie, tels qu'ils apparaissent à la suite de la conférence du bicentenaire de l'Université de Princeton. La présentation date de juillet 1947 et tient compte de discussions et résultats postérieurs à la conférence. Les problèmes sont groupés sous les rubriques suivantes: espaces lenticulaires, rétractes et connexion locale, groupes d'homotopie, groupes d'homotopie des sphères, classes d'homotopie, espaces fibrés, théorie de l'homologie, relations entre homogie et homotopie, groupes de transformations.

*H. Cartan* (Paris).

**Areškin, G. Ya. Structures of locally bicompact $T_1$- and $T_2$-spaces.** Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 213–220 (1949). (Russian)

This paper comprises detailed proofs and extensions of results announced earlier [Doklady Akad. Nauk SSSR (N.S.) **59**, 629–630 (1948); these Rev. **9**, 455]. The principal theorem stated in the review cited is proved by the familiar method of constructing all maximal sets in the lattice $S$ not containing 0 and closed under the operation $\bigcap$; this family of sets is then topologized in an appropriate manner. Analogous results are proved for Hausdorff spaces, and the following very simple characterization is obtained for (bi-) compact $T_1$-spaces: a lattice $S$ is lattice-isomorphic to a closed basis for a compact $T_1$-space if and only if (1) $S$ is distributive and contains zero and unity elements; (2) if $0\neq b<a$, then there exists an element $c$ such that $0\neq c<a$ and $b\bigcap c=0$. *E. Hewitt* (Seattle, Wash.).

**Areškin, G. Ya. On continuous mappings of compact spaces.** Mat. Sbornik N.S. **24**(66), 493–499 (1949). (Russian)

The principal result proved in this paper is the following. Let $X$ and $Y$ be compact metric spaces with open bases $\{U\}$ and $\{V\}$, respectively, such that both $\{U\}$ and $\{V\}$ contain the entire space and are closed under the formation of finite unions of sets. Then $X$ is homeomorphic to $Y$ if and only if there exists a mapping $V=\varphi(U)$ carrying $\{U\}$ onto $\{V\}$ such that $\varphi(U_1\bigcup U_2)=\varphi(U_1)\bigcup\varphi(U_2)$ and such that $\varphi(U)=Y$ if and only if $U=X$. *E. Hewitt.*

**Rauhvarger, I. L. On mappings onto compacta.** Doklady Akad. Nauk SSSR (N.S.) **66**, 13–15 (1949). (Russian)

Let $X$ be a topological space. If there exists a one-to-one continuous mapping of $X$ onto a compactum (compact metric space), then $X$ is said to be contractible onto a compactum. The author is concerned with the conditions under which a separable metric space $X$ can be contracted onto a compactum. [This question has been studied for more general spaces by the reviewer; see Bull. Amer. Math. Soc. **55**, 421–426 (1949); these Rev. **10**, 616.] The following theorems are proved. (I) If the separable metric space $X$ can be contracted onto the compactum $Z$ by means of a mapping $f$, then there exists a compactum $Y$ containing $X$ as a dense subspace such that $f$ can be continuously extended over $Y$. (II) If $X$ can be imbedded in a compactum $Y$ by the addition of a countable set of points, then $X$ is contractible onto a compactum. (III) If $X$ is a dense subset of a compactum $Y$, and $Y\bigcap X'$ is a zero-dimensional set of type $F_\sigma$, then $X$ is contractible onto a compactum. (IV) A locally connected separable metric space, each component of which contains only a finite number of points of non-compactness, is contractible onto a compactum. Two other analogous theorems are proved. *E. Hewitt.*

**Begle, E. G. A note on $S$-spaces.** Bull. Amer. Math. Soc. **55**, 577–579 (1949).

The author proves that if $A$ and $B$ are $S$-spaces (Hausdorff spaces in which each open covering has a star-finite refinement), and if further one of $A$ and $B$ is (bi-)compact, then $A\times B$ is an $S$-space and $\dim(A\times B)\leqq\dim A+\dim B$. Here $\dim(X)\leqq n$ means that each open covering of $X$ has a star-finite refinement of order not exceeding $n+1$. The case in which both $A$ and $B$ are compact was established by Hemmingsen [Duke Math. J. **13**, 495–504 (1946); these Rev. **8**, 334]. It is noted that an example of Sorgenfrey [same Bull. **53**, 631–632 (1947); these Rev. **8**, 594] shows that the compactness condition cannot be wholly omitted. [The author wishes to point out that, to make the mapping $\zeta$ continuous, its definition as given on p. 578, line 12*, should be replaced by the following: if $p\varepsilon T_i$, and $p=q+tr$, where $q\varepsilon R_i$, $r\varepsilon R_{i+1}$, and $0\leqq t\leqq 1$, then

$$\zeta(p\times b)=p\times\{\zeta_i(b)+t\pi_{i+1}\zeta_{i+1}(b)\},$$

the additions referring to barycentric coordinates in single simplexes.] *A. H. Stone* (Manchester).

**Floyd, E. E. The extension of homeomorphisms.** Duke Math. J. **16**, 225–235 (1949).

Assume as a standing hypothesis that $P$ is an open set in a compact metric space $A$, and $Q$ is a subset of a compact metric space $B$, while $f$ is a continuous map from $P$ to $Q$ such that $f|P$ is a homeomorphism of $P$ onto $Q$. For any

set $H$ denote by $H^*$ the set $\bar{H}-H$. Then $Q$ is said to be regular relative to a collection $G$ of closed subsets of $Q^*$, if for each subset $M$ of $Q$ with $M^*$ contained in an element $Y$ of $G$, and for each point $x$ of $Q^*-Y$ there is a neighborhood $V$ in $B$ containing $x$ such that $VQ$ lies in a single component of $Q-M$.

The author proves: (i) If $G$ is a collection of closed subsets of $Q^*$ such that $Q$ is regular relative to $G$, then the map $f|P^*$ from $P^*$ onto $Q^*$ is nonalternating relative to $G$. (ii) Let $G$ be a collection of closed subsets of $Q^*$, and $f^{-1}(G)$ the collection of closed sets $f^{-1}(Y)$, $Y \varepsilon G$. Suppose $P$ is regular relative to $f^{-1}(G)$. Then a necessary and sufficient condition that $Q$ be regular relative to $G$ is that $f|P^*$ be nonalternating relative to $G$. (iii) A necessary and sufficient condition that $Q$ be uniformly locally connected is that $Q$ be locally connected and regular relative to the collection of all closed subsets of $Q^*$. (iv) If $Q$ is uniformly locally connected then $f|P^*$ is monotone. If $P$ is uniformly locally connected, then a necessary and sufficient condition that $Q$ be also is that $f|P^*$ be monotone. (v) If $Q$ is a region on an $n$-sphere, $n \geqq 1$, then $f|P^*$ is nonalternating. (vi) If $P$ is ulc$^{n-1}$ and $Q$ is ulc$^n$, then each $f^{-1}(y)$, $y \varepsilon Q^*$, is $n$-acyclic. (vii) If $P$ is ulc$^n$, then a necessary and sufficient condition that $Q$ be ulc$^n$ is that every $f^{-1}(y)$, $y \varepsilon Q^*$, be $n$-acyclic.

Without the standing hypothesis the author obtains the following theorem. Let $C$ and $D$ be boundary curves and $g(C)=D$ continuous. Then a necessary and sufficient condition that $g$ be nonalternating is that there exist regions $P$ and $Q$ on the 2-sphere $S$ and a continuous map $f$ from $\bar{P}$ to $\bar{Q}$ such that $f|P$ is a homeomorphism from $P$ onto $Q$ and such that $f|P^*$ and $g$ are topologically equivalent.

*D. W. Hall* (College Park, Md.).

**Pontryagin, L. S. Vector fields on manifolds.** Mat. Sbornik N.S. **24(66)**, 129–162 (1949). (Russian)

This paper gives the proofs of results partially announced in a note [C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 34–37 (1942); these Rev. **4**, 147], and also in a paper upon which the present one depends [Mat. Sbornik N.S. **21(63)**, 233–284 (1947); these Rev. **9**, 243]. One is referred to that paper, or the review, for the definitions of the functions $\chi$ and related $\omega$, the space $H(k, l)$ and the pseudomanifold $Z(\omega)$, also denoted by $Z_\chi$, and the characteristic cycles $X_\chi$ associated with the functions $\chi$.

Let $V(x)$ denote a sequence of vector fields $v_1(x), \cdots, v_m(x)$ defined on a differentiable manifold $M^k$ imbedded with continuously turning tangent space in a convenient Euclidean space $R^{k+l}$. Let $\sigma(h)$, $h=0, 1, \cdots, k-1$, be a positive monotonic nondecreasing integral-valued function subject to the condition that $k-1+\sigma(k-1) \leqq m$, the number of fields in $V(x)$. The sequence $V(x)$ is said to be subordinate to the function $\sigma$ at a point $x=a$ if, for every $h$, the system of vectors $v_1(a), \cdots, v_{h+\sigma(h)}(a)$ has a rank not exceeding the number $h$. If such a point $a$ exists, it is called a singular point of type $\sigma$ for the sequence $V(x)$. A special case of this is the more usual notion of a singular point, namely a point $x=a$ at which the vectors $v_1(a), \cdots, v_m(a)$ are not linearly independent. This is obtained for the special functions $\sigma(0)=\cdots=\sigma(q-2)=0, \sigma(q-1)=\cdots=\sigma(k-1)=1$, for some integer $q \leqq k$.

The set of all singular points of a given type $\sigma$, denoted by $F_\sigma$, is a closed set and [theorem 1] is the carrier of the characteristic cycle $X_\chi$ associated with the function $\chi$ defined through $\sigma$ by $\chi(i)=\sigma(k-i)$. By way of converse, if a characteristic cycle is given on an $M^k$ and a cycle $X$ homologous to it is realized in some simplicial decomposition of $M^k$, then [theorem 2] it is possible to construct a sequence $V(x)$ whose singular set $F_\sigma$ for the associated function $\sigma$ consists of the points of $M^k$ belonging to simplexes making up the cycle $X$ (to within a certain well-defined point-set of lower dimension than $k-r$, the dimension of $X$).

When a singular point $a$ is isolated in a (small) simplex $E^r$ there is defined for it (and the sequence $V(x)$) a multiplicity of singularity, denoted by $I_\sigma(V(x))$. There is a considerable discussion of this "index" and of its effective calculation. The definition will not be given here; it depends on a certain mapping of $E^r$ into the space $H(k, l)$ and on the pseudomanifold $Z(\omega)$, and has the form of an incidence number of two spaces which have only $a$ in common. In the case that $Z(\omega)$ is not orientable, the multiplicity is a mark modulo two rather than a signed integer. When the set of singular points of type $\sigma$ makes up a polyhedron $L$ of dimension $k-r$, and each of its simplexes is differentiably disposed in $M^k$, there is defined a cycle of singularity $X_\sigma'$. In this cycle the simplexes $E^{k-r}$ of $L$ appear with a coefficient $X_\sigma'(E^{k-r})$. This is determined as the multiplicity of singularity mentioned above, associated now with some point $a$ of $E^{k-r}$ and some simplex $E^r$ of $M^k$ meeting $E^{k-r}$ in $a$ alone, the point $a$ being interior to both simplexes. The crucial result here is [theorem 3] that the cycle of singularity is homologous to the characteristic cycle (for the same function). This result is also an instrument for the construction of characteristic cycles from appropriate systems of vector fields. One of the characteristic cycles is identified [in theorem 4] with the Euler characteristic of the manifold.

In § 3 it is supposed that one oriented differentiable manifold is differentiably mapped upon another: $f(M^k) \subset P^m$. The critical set of type $p$, $p<k$ and $p<m$, is defined and, in the case that this set is a polyhedron $L$ of $M^k$ there is defined the cycle of singularity or critical cycle. In the case that $P^m$ is a Euclidean space it is proved [theorem 5] that the critical set carries the characteristic cycle (for an appropriate function). Further, that cycle is homologous to the critical cycle, when the latter is defined.

Theorem 6. For a closed oriented differentiable manifold $M^k$, the characteristic cycle $X_{1k}$ is always equal to zero. This cycle is associated with the function $\chi(1)=k$, and $\chi$ vanishes elsewhere. Finally, theorem 7 states that, for an arbitrary four-dimensional oriented differentiable closed manifold $M^4$, the characteristic cycles $X_{41}$ (the Euler characteristic) and $X_{22}$ [which is new: see the preceding paper] are connected by the relation $X_{41} \sim X_{22} \pmod 2$.

*L. Zippin* (Flushing, N. Y.).

**Pontryagin, L. S. Some topological invariants of closed Riemannian manifolds.** Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 125–162 (1949). (Russian)

This is the detailed exposition of results announced earlier [C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 91–94 (1944); these Rev. **6**, 183]. The author states that "the paper, in so far as possible, translates into the language of differential forms the contents of an earlier work" [Mat. Sbornik, N.S. **21(63)**, 233–284 (1947); these Rev. **9**, 243]. A somewhat general introduction is followed [in § 1] by an exposition of the tensor theory of homology on manifolds. In § 2 the author treats the space $H(k, l)$ of $k$-dimensional planes through the origin of a Euclidean $R^{k+l}$ as a homogeneous space under the action of the group $G$ of proper orthogonal transformations of $R^{k+l}$ into itself, the origin being fixed. Here he derives all closed invariant fields over

$H(k, l)$ of order not exceeding $k$, on the assumption that $l \geqq k+1$. These fields give rise [in § 3] to the "characteristic fields" on a manifold $M^k$, provided that $M^k$ is mapped into $R^{k+l}$ regularly (in the sense of Whitney). It is shown that the characteristic fields may be expressed in terms of the Riemann metric tensor associated with $M^k$ through its regular imbedding in $R^{k+l}$; but the fields are independent (to within homology) of the particular regular imbedding, and independent also of the number $l$. To these fields (of order $r$) there correspond [in § 4] certain earlier defined characteristic cycles (of dimension $k-r$). Thus, if $P$ denotes the field, $X(P)$ the associated cycle, and if $Y$ is an $r$-dimensional integral-coefficient cycle on $M^k$, the correspondence is given by the fact that the incidence number of the cycle $X(P)$ and the cycle $Y$ is measured by the integral over $Y$ of the field $P$: $\int P = I\{X(P), Y\}$. The cycle $X(P)$ may be expressed linearly in terms of certain basic characteristic cycles, and the remainder of the paper [§§ 4 and 5] is concerned with integral formulas for the coefficients of this expansion. Here the author also derives the Gauss-Bonnet formula [reference to Chern, Ann. of Math. (2) 45, 747–752 (1944); these Rev. 6, 106, but none to Chern, Proc. Nat. Acad. Sci. U. S. A. 30, 269–273 (1944); these Rev. 6, 106, which is closely related to the present paper].

*L. Zippin* (Flushing, N. Y.).

**Bassi, Achille. Sopra l'esistenza di una varietà topologica con numeri del Betti assegnati.** Anais Acad. Brasil. Ci. 21, 69–73 (1949).

In an earlier paper [Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 6, 669–714 (1935)] the author showed that for $n=4, 8$ and for $n \neq 0 \bmod 4$, there exists an $n$-dimensional compact orientable topological manifold with preassigned Betti numbers $R_i$, provided $R_0 = R_1 = 1$ and $R_i = R_{n-i}$ and, in case $n = 4p+2$, $R_{2p+1}$ is even. He now announces that this result holds if $n = 4p$ provided the Euclidean space $E_{2p}$ with one point removed is a topological loop (presumably of a special sort). *P. A. Smith* (New York, N. Y.).

**Hu, Sze-tsen. On singular homology in differentiable spaces.** Ann. of Math. (2) 50, 266–269 (1949).

It was shown by Eilenberg [same Ann. (2) 48, 670–681 (1947); these Rev. 9, 52] that the singular homology theory of a differentiable manifold $M$ of class $C^3$ could be based on singular simplexes of class $C^3$. A much simpler proof of this fact is given here. It avoids using the existence of a triangulation of class $C^3$. It is assumed that $M$ is a subset of Euclidean space and is a neighborhood retract there by means of a function of class $C^3$.

*N. E. Steenrod* (Princeton, N. J.).

**Eckmann, B., Samelson, H., and Whitehead, G. W. On fibering spheres by toruses.** Bull. Amer. Math. Soc. 55, 433–438 (1949).

It is shown that the $n$-sphere can be represented as a fibre bundle where the fibre is an $s$-dimensional torus $T^s$ if and only if $n$ is odd and $s=1$; and for no $n$ and $s$ can Euclidean $n$-space be represented as a bundle with fibre $T^s$. The nonexistence is proved allowing the group of $T^s$ to be any group of homeomorphisms. In the case of the $n$-sphere, the base space $M$ is required to be separable metric [this is no restriction since it can be proved]. In the case of Euclidean $n$-space, $M$ is assumed to be a locally finite polyhedron. The proof involves an application of the result of Eilenberg and MacLane [Ann. of Math. (2) 46, 480–509 (1945); these Rev. 7, 137] that, if $\pi_i(M) = 0$ for $i=1$ and $2 < i < n$, then $\pi_2(M)$ determines the homology groups $H_i(M)$ for $i < n$.

*N. E. Steenrod* (Princeton, N. J.).

**Ganea, Tudor. Sur les espaces de recouvrement des rétractes.** C. R. Acad. Sci. Paris 228, 1470–1472 (1949).

This note establishes certain relations between the fundamental group of a space and the fundamental group of a retract. Earlier results on this problem have been obtained by Borsuk [Fund. Math. 21, 91–98 (1933)]. The definition of fundamental group used in the one given by Chevalley [Theory of Lie Groups, v. 1, Princeton University Press, 1946; these Rev. 7, 412]. Some of the results stated are as follows. Let $E$ be a space with a simply connected covering space $(E, f)$ and let $A$ be a retract of $E$. Then $\Pi_1(A)$ is the homomorphic image of $\Pi_1(E)$. A necessary and sufficient condition that $\Pi_1(E)$ be isomorphic to $\Pi_1(A)$ is that $f^{-1}(A)$ is connected. If $E$ is a topological group then $\Pi_1(A)$ is Abelian. *D. Montgomery* (Princeton, N. J.).

**Volpato, Mario. Sull'esistenza di punti uniti nelle trasformazioni univoche e continue del cerchio.** Ann. Mat. Pura Appl. (4) 27, 101–105 (1948).

Let $G$ be a simple closed curve $g$ in $E^2$ plus its interior. Let $t$ be a topological mapping $G \to E^2$ such that $t(g)$ contains interior points of $G$. Extending an earlier result of Scorza Dragoni [Ann. Mat. Pura Appl. (4) 25, 43–65 (1946); these Rev. 9, 455] the author derives sufficient conditions for the existence of a fixed point of $t$ in terms of the effect of $t$ upon the arcs into which $g$ is separated by $t(g)$. The proof depends on a preliminary theorem concerning the fixed points of a transformation operating only on $g$. *P. A. Smith.*

**Rado, R. Factorization of even graphs.** Quart. J. Math., Oxford Ser. 20, 95–104 (1949).

A graph $\Gamma$ is locally finite if each node is incident with only a finite number of edges. A factor of $\Gamma$ is a set $F$ of edges of $\Gamma$ such that each node of $\Gamma$ is incident with just one member of $F$. A loop of $\Gamma$ is an edge incident with just one node. A graph is even if its set of nodes can be partitioned into two mutually exclusive classes $A$ and $B$ such that no two distinct nodes of either class are joined by an edge. Thus an even graph may have loops. A set $S$ of nodes of a graph is independent if no two nodes of $S$ are joined by an edge. P. Hall has shown that a finite even graph $\Gamma$ possesses a factor if and only if, given any independent set of any number $k$ of nodes $a_x$, there are $k$ distinct nodes each of which is joined by an edge with at least one of the nodes $a_x$ [J. London Math. Soc. 10, 26–30 (1935)]. The author shows that this criterion is valid for all locally finite graphs. He restates his theorem, and gives his proof, in terms of systems of abstract aggregates. *W. T. Tutte* (Toronto, Ont.).

**Shannon, Claude E. A theorem on coloring the lines of a network.** J. Math. Physics 28, 148–151 (1949).

The author considers a finite graph without loops, i.e., in which each edge has two distinct ends. Two nodes may be joined by more than one edge. He shows that the edges can be coloured in $[\frac{3}{2}m]$ or fewer colours so that no two edges of the same colour have a common end, where $m$ is the maximum number of edges meeting at a node. Simple constructions are given for graphs, corresponding to any desired value of $m$, for which $[\frac{3}{2}m]$ colours are necessary.

*W. T. Tutte* (Toronto, Ont.).

## GEOMETRY

***Segre, Beniamino. Lezioni di Geometria Moderna. Vol. I. Fondamenti di Geometria Sopra un Corpo Qualsiasi.** Nicola Zanichelli, Bologna, 1948. iv+195 pp. 1200 Lire.

The first part of this book is an introduction (of less than 80 pages) to abstract algebra, especially the ideas of groups, rings and, above all, of fields. The presentation, which is completely lucid and yet avoids surfeiting the reader with detail, has clearly been carefully planned. It begins with residue classes modulo an integer and goes on from there to introduce more abstract ideas. The discussion of fields of characteristic $p$ is especially impressive, and provides much information not easily accessible elsewhere.

The second part deals no less attractively with the foundations of linear projective geometry in an arbitrary field. The definition of the cross-ratio of four points as a class of conjugate elements of the base field (coinciding with the usual definition for commutative fields) is noteworthy among many points of interest. The first two chapters of this part discuss projective spaces defined analytically and synthetically, and the place of the theorems of Desargues and Pappus in the theory. The next two chapters on Pascal spaces (in which Pappus' theorem holds) deal with subspaces and Grassmann coordinates and isomorphisms of various types (homography, collineation, involution, polarity, etc.) between spaces. There is a final chapter on finite spaces. It is natural to compare this book with the recent volume by Hodge and Pedoe [Methods of Algebraic Geometry, v. 1, Cambridge University Press; Macmillan, New York, 1947; these Rev. 10, 396]. The slight difference of approach (Segre being interested in more abstract fields) has ensured that these two books complement rather than compete with one another, and no reader of either should be without both.

The standards of rigour and accuracy achieved in this work are such as all mathematicians (and not merely algebraic geometers steeped in the traditions of the Italian school) will appreciate. It is to be followed by two further volumes which will be looked forward to with interest. The first of these is to deal with nonlinear projective geometry and its arithmetical applications, and the second with birational geometry. *D. B. Scott* (London).

***Coxeter, H. S. M. The Real Projective Plane.** McGraw-Hill Book Company, Inc., New York, N. Y., 1949. x+196 pp. $3.00.

The chapter headings in this book are as follows. 1. A comparison of various kinds of geometry; 2. Incidence; 3. Order and continuity; 4. One-dimensional projectivities; 5. Two-dimensional projectivities; 6. Conics; 7. Projectivities on a conic; 8. Affine geometry; 9. Euclidean geometry; 10. Continuity; 11. The introduction of coordinates; 12. The use of coordinates; Appendix I. The complex projective plane; Appendix II. Bibliography; Index. The scope of the work is that of an elementary account of plane projective geometry in the real field, and the treatment is synthetic and axiomatic. It is a novel feature to find the axiomatic treatment of order and continuity worked into a text of as elementary a character as this. That the author has succeeded in keeping the treatment so simple as to make the book suitable for an introductory course in the subject is due in part to the great care with which the axioms of order and continuity have been selected, and in part to the very careful nature of the exposition. Among the novelties may be mentioned the introduction of the conic as the locus of points incident with their corresponding lines in a hyperbolic polarity (from which the self-dual character of the conic is evident). The whole treatment, especially of the applications of the axioms of order, is very elegant. The diagrams are numerous and well-drawn, and there are a number of exercises for the student using the book.

*J. A. Todd* (Cambridge, England).

**Sperner, Emanuel. Beziehungen zwischen geometrischer und algebraischer Anordnung.** Arch. Math. 1, 148–153 (1948).

Summary of an unpublished paper. The order-relations in a "space" with two types of elements (points $\alpha$ and hyperplanes $h$) may be described by an order function $h(\alpha)$ defined for any $\alpha$ and $h$ and capable of the values $\pm 1$ and 0 only. Incidence is expressed by $h(\alpha)=0$; $h(\alpha)h(\beta)=-1$ means that $h$ lies between $\alpha$ and $\beta$; and $h, k$ and $\alpha, \beta$ separate each other if and only if $h(\alpha)h(\beta)k(\alpha)k(\beta)=-1$ [see E. Sperner, Arch. Math. 1, 9–12 (1948); these Rev. 10, 138]. In the case of a projective space $P_n$ over a field $K$ (which may be skew) the order functions may be related to the semiorderings of $K$ (i.e., the divisions of the nonzero elements of $K$ into two classes of positive and negative elements satisfying the monotonic law of multiplication: $ab>0$ if and only if $a, b$ are of the same sign). The detailed investigation of the relations between the geometrical order functions and the algebraic semiorderings are based on the following two results. (1) If a semiordering of $K$ is given, and if for every point $\alpha$ and every hyperplane $h$ a definite $(n+1)$-tuple of coordinates $u_i$, $x_i$ is chosen then the definition $h(\alpha)=+1$, $-1$, $0$ according to whether $\sum u_i x_i>0$, $<0$, $=0$ yields an order function satisfying the "hyperplane relation": $h(\alpha)h(\beta)k(\alpha)k(\beta)=1$ for any $h, k$ which are not incident with $\alpha, \beta$, but are both incident with one and the same point on a straight line through $\alpha, \beta$. (2) Conversely, an order function satisfying the hyperplane relation determines a semiordering, which may be defined by making the cross-ratio $D(h, k|\alpha, \beta)=(\sum u_i x_i)^{-1}(\sum u_i y_i)(\sum v_i y_i)^{-1}(\sum v_i x_i)$ positive or negative according to whether $h(\alpha)h(\beta)k(\alpha)k(\beta)$ is $+1$ or $-1$. *F. A. Behrend* (Melbourne).

**Haupt, Otto. Schwache Ordnung im projektiven $n$-dimensionalen Raum $R_n$.** Math. Ann. 120, 473–491 (1949).

In previous papers [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1941, 57–66; Ann. Mat. Pura Appl. (4) 23, 123–148 (1944); these Rev. 8, 164; 7, 468] the author has defined and studied comparatively the strong and the weak component orders (abbreviated SC and WC) and the strong and the weak point orders (abbreviated SP and WP) for a continuum of a Euclidean space with respect to systems of order characteristics consisting of hyperplanes. In this paper the space is the $n$-dimensional projective space $R_n$; the $t$-dimensional projective subspaces are denoted by $L_t$; the order characteristics are all hyperplanes $L_{n-1}$; the investigated set is a continuum $K$ of $R_n$. The principal results are as follows. If $\mathrm{WP}(K)<\infty$, $K$ is a hereditary arc sum (this projective proposition is proved by reduction to the Euclidean one previously established). If $\mathrm{WC}(K)=p<\infty$, then $\mathrm{SC}(K)<3p+1$. If $\mathrm{WP}(K)=p<\infty$, and $\mathfrak{A}$ denotes the system of the $L_{n-1}$ through the axis $A=L_{n-t}(2\leqq t\leqq n)$, there

exist for $t=2$ (at most) enumerably many $L_{n-1}$, the intersection of which with $K-K\cdot A$ is punctual; in the general case $t>2$, the $L_{n-1}$ having the latter property are dense in $\mathfrak{a}$. The tools used in the proofs are borrowed from the topological (Menger's) theory of curves. The "bridge theorem" [Brückensatz; see Haupt, Nöbeling, and Pauc, J. Reine Angew. Math. 182, 105–121 (1940); these Rev. 2, 130] is completed under the assumption of local connectedness for the continuum.

The fundamental configuration appears as follows. The intersection $D$ of $K$ with a fixed $H=L_{n-1}$ is decomposed into a finite number of closed subsets $T_j$ $(j=1,\cdots,k)$ called "patches" (Stücke) of $D$, which eventually specialize into components of $D$ or points. Each $T_j$ is surrounded by a neighbourhood $U_j$ such that the closures $\bar{U}_{j_1}$ and $\bar{U}_{j_2}$ of $U_{j_1}$ and $U_{j_2}$ for $j_1\neq j_2$ do not intersect for $j_1\neq j_2$. The boundary of $U_j$ is denoted by $B_j$. If $K\cdot B_j$ is not empty (or in other words, if $K$ is not contained in $U_j$) and $\bar{U}_j$ is Euclidean (that is, there exists a hyperplane not intersecting $\bar{U}_j$), every closed set $F_j$ contained in $B_j$ and containing $K\cdot B_j$ with $H\cdot F_j=0$ either lies in $\bar{U}_j$ on one side of $H$, or is the sum of two nonempty closed sets $F_j^1$, $F_j^2$ lying in $\bar{U}_j$ on different sides of $H$; $F_j$ in the first case, $F_j^1$ and $F_j^2$ in the second case are called "lids" (Deckel) of $U_j$. Using these notions the author defines the "Schnittstücke (Schnittkomponenten, Schnittpunkte)" of first and second type. The investigation is focused on the properties of intersection of $K$ with hyperplanes $L'_{n-1}$ in a neighbourhood of a fixed hyperplane $H$. If for instance there exists in $U_j$ a bridge linking two $U_j$-lids, this bridge is intersected by the $L'_{n-1}$ sufficiently near to $H$.

The author gives elaborate definitions throughout his paper. The proofs are worked out carefully and in detail.

*C. Y. Pauc* (Cape Town).

**Cuesta, N. Structures and their automorphisms.** Revista Mat. Hisp.-Amer. (4) 8, 277–282 (1948). (Spanish)

The author shows that there exist two different ordering relations (of different order types) on a certain set such that the automorphisms of the two relations are the same and are infinite in number. He proposes the question: Are there two projective spaces, with base fields of different types, for which the groups of automorphisms of the projective systems are isomorphic? The significance of these considerations for the Erlangen Programm is discussed.

*P. M. Whitman* (Silver Spring, Md.).

**Narasinga Rao, A. Geometry as the study of space-structure.** Math. Student 15 (1947), 89–92 (1949).

**Hesselbach, B. Über vierdimensionale Drehungen.** Math. Ann. 121, 22–32 (1949).

When points in Euclidean 4-space are represented by quaternions, the general direct orthogonal transformation $x\to z$ takes the form $z=axb$ [see, e.g., Coxeter, Amer. Math. Monthly 53, 136–146 (1946); these Rev. 7, 387]. It is thus the product of left and right Clifford translations $z=ax$ and $z=xb$. The author finds expressions for these, using general affine coordinates instead of Cartesian coordinates. In terms of the matrix $A$ of a given orthogonal transformation $z^i=\sum h_k{}^i x^k$, whose characteristic equation is $\omega^4+a_1\omega^3+a_2\omega^2+a_3\omega+1=0$, he defines numbers $\lambda$ and $\rho$ such that $4\lambda\rho=-a_1$, $4(\lambda^2+\rho^2)=a_2+2$; and he shows that the matrices of the component Clifford translations are $L$ and $R$ (so that $LR=RL=A$), where

$$4(\lambda^2-\rho^2)L=-\lambda(A^2-A^{-2})+2\rho(2\lambda^2-1)(A-A^{-1})+4\lambda(\lambda^2-\rho^2)I,$$

and $R$ is given by the same formula with $\lambda$ and $\rho$ interchanged.

*H. S. M. Coxeter* (Toronto, Ont.).

**Rios de Souza, Jayme. Geometric representation of complex systems.** Anais Fac. Ci. Pôrto 31, 65–79 (1946). (Portuguese)

Some remarks about the notion of points at infinity for $n$-dimensional complex Euclidean space and the corresponding "Segre variety."

*L. Nachbin* (Chicago, Ill.).

**Bagchi, Haridas. Note on the common tangents of a cubic and one of its sextactic conics.** Bull. Calcutta Math. Soc. 40, 135–139 (1948).

The twelve contacts of a plane nonsingular cubic with those of its tangents which touch a general conic lie on a quartic curve. Among the many limiting cases of this result, that in which the conic becomes a sextactic conic gives a set of six residual points of contact which lie on a conic and are joined in pairs by three lines through an inflexion.

*J. G. Semple* (London).

**Bagchi, Haridas. Circles of double contact of a bi-circular quartic.** Bull. Calcutta Math. Soc. 40, 207–225 (1948).

This paper consists of a study of the systems of contact circles of bicircular quartics and circular cubics by the methods of plane coordinate geometry. Among the results of section I which the author claims to be new are certain properties of the evolute $\Omega$ of a bicircular quartic $\Gamma$, as, for example, that the 16 cusps of $\Omega$ lie by fours on the four focal conics of $\Gamma$, and the corresponding cuspidal tangents are the common tangents of these conics and the associated circles of inversion of $\Gamma$. Section II deals with bicircular quartics which possess one or more points of undulation. Section III deals with the simplest irrational forms of the equation of a bicircular quartic. Section IV deals with circular cubics and extends to them the results of sections I and III.

*J. G. Semple* (London).

***Yar, Ali. Sur la forme associée d'un cercle de l'espace.** Université d'Istanbul. Faculté des Sciences. Recueil de mémoires commémorant la pose de la première pierre des Nouveaux Instituts de la Faculté des Sciences, pp. 1–8, Istanbul, 1948.

L'auteur considère un cercle comme l'intersection des deux sphères

$$\alpha_i(x^2+y^2+z^2)+2\beta_i x+2\gamma_i y+2\delta_i z+\epsilon_i=0$$

$(i=1, 2)$, et étudie les 10 mineurs du tableau

$$\left\|\begin{matrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \epsilon_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \epsilon_2\end{matrix}\right\|.$$

Il exprime la condition, nommée "forme associée du cercle" q'une droite rencontre le cercle. Les coordonnées d'un cercle sont évidemment analogues aux coordonnées pluckériennes dans l'espace à quatre dimensions.

*O. Bottema* (Delft).

**Davatz, W. Einführung der Enden als Parametergrössen.** Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. 7, 13–30 (1941).

The author describes the hyperbolic plane in terms of point coordinates $(\alpha, \beta, \gamma)$ and line coordinates $[u, v, 1]$

with the absolute conic $4\alpha\gamma-\beta^2=0$ or $u-v^2=0$. He obtains the conditions for one point to lie between two others, and a parametric expression for the matrix of the general congruent transformation. Similar results were obtained more neatly by Klein in his Vorlesungen über nicht-euklidische Geometrie [3d ed., Springer, Berlin, 1928].

*H. S. M. Coxeter* (Toronto, Ont.).

**Reifenberg, E. R. A problem on circles.** Math. Gaz. **32**, 290–292 (1948).

Recently Besicovitch proved the following theorem [Proc. Cambridge Philos. Soc. **41**, 103–110 (1945); these Rev. **7**, 10]. Let $C$ be the unit circle. Then any set of circles satisfying the following conditions: (1) each circle intersects $C$; (2) each circle has radius at least 1; (3) no circle contains the origin or the center of any other circle, has less than 22 members. The author proves that such a set of circles has at most 18 members, and that such a set of 18 circles exists (thus 18 is best possible). *P. Erdös.*

**Fejes Tóth, L. On the densest packing of spherical caps.** Amer. Math. Monthly **56**, 330–331 (1949).

The author gives a new proof of the following result. Let there be given $n$ points on the surface of the unit sphere. Then there always can be found two of them whose spherical distance is not greater than (1) $d\leqq\arccos\{\frac{1}{2}(\cot^2\omega-1)\}$, where $\omega=\frac{1}{2}n\pi/(n-2)$. The author uses (1) to sharpen a result of [illegible]. *P. Erdös.*

**Bouvaist, Robert, et Thébault, Victor. Cubiques gauches et points remarquables associés au tétraèdre.** C. R. Acad. Sci. Paris **228**, 1689–1691 (1949).

**Thébault, Victor. Sphères remarquables associées à un polygone gauche.** Bull. Soc. Roy. Sci. Liége **16**, 149–162 (1947).

**Thébault, Victor. Sur un groupe de paraboles associées à un triangle.** Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. N.S. **18**, 31–33 (1949).

**Veldkamp, G. R. On the power of some remarkable points of the triangle with respect to remarkable circles of the triangle.** Nieuw Tijdschr. Wiskunde **36**, 245–256 (1949). (Dutch)

**Hebroni, Pessach. On the eight "Feuerbach circles" attached to a triangle.** Riveon Lematematika **3**, 28–32, 37 (1949). (Hebrew. English summary)

**Narayanamurty, T. On a system of circles associated with a symmedian of a triangle.** Math. Student **15** (1947), 101–102 (1949).

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### *Convex Domains, Integral Geometry*

**Baldus, Richard. Minimumeigenschaften $n$-dimensionaler Würfel.** Math. Ann. **120**, 462–472 (1949).

Let $P$ denote an $n$-dimensional parallelepiped in $n$-dimensional Euclidean space $R_n$, $B$ a base of $P$, $e$ the corresponding edge, $h$ the corresponding altitude, and $\sigma_q$ the sum of the $q$-dimensional measures ("volumes") of the $q$-dimensional sides of $P$ ($q=1,\cdots,n$). The equation $P=(B,e)$ means that $P$ is generated from $B$ by the translation $e$. The main theorem in this paper is the following. Among all $n$-dimensional parallelepipeds $P$ with $\sigma_n=$constant, the cube yields the minimum of $\sigma_\nu$ for $\nu=1,\cdots,n-1$. First the cases $n=2$ and $n=3$ are treated completely. In the general case the theorem is assumed for parallelepipeds of dimension less than $n$. The proof for dim $P=n$ rests on the transformations $\mathfrak{S}$ and $\mathfrak{A}$, which both preserve $\sigma_n$ and reduce $\sigma_\nu$ for $\nu=1,\cdots,n-1$ and are defined as follows: If $e$ is not orthogonal to $B$, $\mathfrak{S}$ is the shear $P\to(B,h)$. If $B$ is not an $(n-1)$-dimensional cube and $C$ the cube having the same volume $\mathfrak{A}$: $(B,h)\to(C,h)$.

[Reviewer's note: these elementary extremal properties may be immediately transposed to $n\times n$ matrices with prescribed determinant, using the formula $|A\cdot A'|^{\frac{1}{2}}$ for the volume of the parallelepiped representing the rectangular matrix $A$ in $R_n$ [see G. Birkhoff and S. MacLane, A Survey of Modern Algebra, Macmillan, New York, 1941, p. 294; these Rev. 3, 99].] *C. Y. Pauc* (Cape Town).

**Hadwiger, H. Kurze Herleitung einer verschärften isoperimetrischen Ungleichung für konvexe Körper.** Rev. Fac. Sci. Univ. Istanbul (A) **14**, 1–6 (1949). (German. Turkish summary)

Let $V$ be the volume of a convex solid in ordinary space, $F$ and $M$ the area and the integral of the mean curvature of its surface, $r$ the radius of the inscribed sphere. Minkowski proved that $M^2\geqq 4\pi F$. The author gives a new proof and deduces that $F^3-36\pi V^2\geqq(F^{\frac{1}{3}}-(4\pi)^{\frac{1}{3}}r)^6$. *L. C. Young.*

**Nöbeling, Georg. Über die Hauptformel der ebenen Kinematik von L. A. Santaló und W. Blaschke. I.** Math. Ann. **120**, 585–614 (1949).

**Nöbeling, Georg. Über die Hauptformel der ebenen Kinematik von L. A. Santaló und W. Blaschke. II.** Math. Ann. **120**, 615–633 (1949).

Let $O_i$ ($i=1,2$) be two open point sets in the plane each of which is bounded by either a finite number of distinct Jordan curves (with continuous curvatures) or by a finite number of distinct simple closed polygons. Let their areas be $F_i$, their perimeters $\Lambda_i$, and let $K_i$ and $K_{12}$ be $2\pi$ times the Euler-Poincaré characteristics of triangulations of $O_i$ and of the intersection $O_1\cdot O_2$, respectively. The formula of Santaló and Blaschke reads

$$\int_{O_1\cdot O_2}K_{12}\dot{O}_2=2\pi(K_1\cdot F_2+K_2\cdot F_1+\Lambda_1\cdot\Lambda_2),$$

where the integration is performed over all possible positions of $O_2$ such that it intersects the fixed set $O_1$.

This paper extends the validity of this formula to sets $O$ such that (1) $O$ is open and bounded; (2) the complement $E-O$ consists of only a finite number of components. Now $K(O)$ is defined as $2\pi\cdot[\rho(O)-\sigma(O)]$, where $\rho(O)$ ($\leqq\infty$) is the number of components of $O$ and $\sigma(O)$ ($<\infty$) is the number of bounded components of $E-O$. The Lebesgue area of $O$ is $F$. To define $\Lambda$ let $O^*$ ($=\bar{O}\cdot\overline{E-O}$) be the boundary of $O$, and $O^{**}$ the boundaries of the finite or countably infinite components of $E-\bar{O}$. By definition,

$$\Lambda(O)=L(O^*)+L(O^*-\sum_a O^{a*}),$$

where $L$ is the linear outer measure of Carathéodory. That is, $\Lambda(O)$ is the linear outer measure of $O^*$, where every point not in the boundary of a component of $E-\bar{O}$ is counted twice.

In the first part of the paper it is assumed that $\Lambda_1$ and $\Lambda_2$ are finite. The second part proves the formula in the remain-

ing cases. In part I, the proof is immediate if $\rho(O_1)$ or $\rho(O_2)$ is infinite. Otherwise the proof proceeds by establishing the formula successively when (1) $O_i^*$ are representable as the sum of finitely many segments, (2) $O_i^*$ are representable as the sum of finitely many convex arcs having only end points in common, (3) $O_i^*$ have no isolated points, and (4) $O_i^*$ are arbitrary. *C. B. Allendoerfer* (Haverford, Pa.).

**Santaló, L. A. Integral geometry on surfaces.** Duke Math. J. 16, 361–375 (1949).

The surfaces under consideration are two-dimensional complete analytic Riemannian surfaces having the property that they can be covered by a finite number of domains in which proper polar coordinate systems can be defined. Generalizations of the most important theorems of integral geometry are obtained for such surfaces, and known results for surfaces of constant curvature are derived as applications.

Let $ds^2=g_{\alpha\beta}dx^\alpha dx^\beta$ $(\alpha, \beta=1, 2)$, $F=(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)^{\frac{1}{2}}$, and

$$p_\alpha=\partial F/\partial\dot{x}^\alpha.$$

Then the exterior differential form $dG=[dx^\alpha dp_\alpha]$, taken in absolute value, has suitable invariant properties and is called the density for sets of geodesics. Its integral is the measure of a set of geodesics. This measure is not applicable to surfaces on which all or almost all geodesics are transitive in the sense of G. A. Hedlund [Bull. Amer. Math. Soc. 45, 241–260 (1939)]. Application of this measure is made to sets of geodesics cutting a given curve, to geodesics intersecting a curve contained in a convex domain, and to pairs of intersecting geodesics. The resulting formulae are familiar in more special types of surfaces.

The density of a set of oriented geodesic segments is called the cinematic density $dK$ and has the expression $dK=2[dG\,ds]$. Among the applications of this is the Poincaré formula for surfaces, namely, $\int_{K\cdot C\neq 0} n(K, C)dK=4L_0L$, where $n(K, C)$ is the number of intersections of the (suitably restricted) curves $K$ and $C$, and $L_0$ and $L$ are their respective lengths. Another application is an inequality of the isoperimetric type which is stronger than the known inequalities when specialized to surfaces of constant curvature. It is stated as follows. Given, on the surface $S$, a convex domain of area $F$ bounded by a curve $C$ of length $L$; if $L_0^*$ is the mean length of the geodesic circles of radius $\lambda$ which intersect $C$ and $F+A_\lambda$ is the area bounded by the exterior parallel curve to $C$ at distance $\lambda$, where $\lambda$ lies between the maximum radius of geodesic circles contained in $C$ and the minimum radius of geodesic circles containing $C$, then

$$LL_0^*-\pi(F+A_\lambda)\geqq 0.$$

*C. B. Allendoerfer* (Haverford, Pa.).

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## *Algebraic Geometry*

**Samuel, Pierre. Multiplicités des composantes excédentaires d'intersection.** C. R. Acad. Sci. Paris 228, 158–159 (1949).

Let $U$ and $V$ be two (irreducible) varieties of respective dimensions $u$ and $v$, lying in an $n$-dimensional affine space $A$. It is known that any component $M$ of $U\bigcap V$ is of dimension $m\geqq u+v-n$. The theories of intersection multiplicity of Weil and Chevalley apply to proper components, that is, to the case where equality holds in this dimension relation. The author generalizes this theory to the case of nonproper components.

Let $\Delta$ be the diagonal of the product space $A\times A$, and $M^\Delta$ the subvariety of $\Delta$ whose projection on the first factor is $M$; let $(x_i)$ and $(x_i')$ be independent general points of $U$ and $V$. Then the multiplicity $i(M; U\cdot V)$ of $M$ in $U\bigcap V$ is defined as the multiplicity of the ideal generated by the $x_i-x_i'$ in the quotient ring of $M^\Delta$ on $U\times V$. This latter multiplicity is meant in the sense previously defined by the author [same C. R. 225, 1244–1245 (1947); these Rev. 9, 325]. For proper components this definition coincides with that of Chevalley. A number of theorems about this intersection multiplicity are stated without proof. They involve a criterion for multiplicity 1, a reduction to the case of proper components, multiplicity formulas concerning linear varieties, product varieties, and projections, and a result on birational invariance. This last result states, namely, that $i(M; U\cdot V)$ is invariant under birational transformations of any variety containing $U$ and $V$ provided the transformation is biregular along $M$. The associativity formula need not hold. *I. S. Cohen* (Cambridge, Mass.).

**Samuel, Pierre. Multiplicités des composantes singulières d'intersection.** C. R. Acad. Sci. Paris 228, 292–294 (1949).

[Cf. the preceding review.] In this note the Weil-Chevalley intersection theory is generalized in another direction. Let $\Omega^w$ be a variety in an affine space $A^n$ (superscripts indicate dimension). Let $B$ be a subvariety of $\Omega$ and let $X^{w-s}$ and $Y^{w-t}$ be cycles on $\Omega$ which pass through $B$ and such that multiples of each are complete intersections in the neighborhood of $B$. That is, there exist cycles $X_1^{n-s}$ and $Y_1^{n-t}$ and integers $a$ and $b$ such that $aX-\Omega\cdot X_1$ and $bY-\Omega\cdot Y_1$ do not pass through $B$. The intersection product $X\cdot Y$ is defined locally (at $B$) by the formula $ab\,X\cdot Y=X_1\cdot\Omega\cdot Y_1$; if $B$ is simple for $\Omega$ then this definition reduces to Chevalley's [Trans. Amer. Math. Soc. 57, 1–85 (1945), p. 72; these Rev. 7, 26]. If $X$ and $Y$ are complete intersections at all components of $X\bigcap Y$ and if each component has dimension $w-s-t$, then $X\cdot Y$ is defined globally by linearity. The intersection multiplicity $i_0(M; X\cdot Y)$ is the coefficient of $M$ in $X\cdot Y$; it need not be an integer if $M$ is not simple. The associativity rule holds, and theorems are stated on the behavior of the intersection multiplicity with respect to product, projection, and birational transformation. A definition of specialization of cycle is indicated; under suitable hypotheses, if the cycles $X$ and $Y$ specialize to $X'$ and $Y'$, then $X\cdot Y$ specializes uniquely to $X'\cdot Y'$. There is, finally, a discussion of the group of all cycles on $\Omega$ modulo the subgroup of those which are complete intersections, as well as a discussion of the corresponding group in a neighborhood of a subvariety of $\Omega$. *I. S. Cohen.*

***Du Val, Patrick. Removal of singular points from an algebraic surface.** Université d'Istanbul. Faculté des Sciences. Recueil de mémoires commémorant la pose de la première pierre des Nouveaux Instituts de la Faculté des Sciences, pp. 21–25, Istanbul, 1948.

We quote from the author's introduction: "Geometric proofs [that any algebraic surface can be birationally transformed into one without any singular points] have been attempted, . . . none of which has escaped criticism. The most important of these are due to Albanese [Rend. Circ. Mat. Palermo 48, 321–332 (1924)] and Severi [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) $23_2$, 527–539 (1914)]. The former shows, by a method acknowledged to be perfectly sound, that any algebraic surface can be trans-

formed into one having no point of multiplicity higher than 2. . . . Severi introduces the important distinction between proper and improper singularities, a proper singularity being such that every hyperplane section of the surface containing it has lower genus than the general hyperplane section. He . . . shows . . . that a surface without proper singularities can be transformed into one without singularities of any kind. This . . . part of his work is generally admitted to be sound. What I propose to prove is therefore that: An algebraic surface without points of multiplicity greater than 2 can be birationally transformed into one without proper singularities. Thus by combining the present result with the sound parts of Albanese's and Severi's investigations, a proof of the main theorem is obtained."

The proof of the theorem consists of three parts. (1) The surface $F$, having no singularities of multiplicity greater than 2, is projected into a multiple plane $F'$. By properly choosing the center of projection this can be done so that the proper singularities of $F'$ are the projections of those of $F$, and are singular points of the double curve of $F'$ at which two sheets are permuted. (2) By a succession of quadratic transformations, $F'$ is transformed into a multiple surface on which the proper singularities are ordinary double points of the double curve at which two sheets are permuted. A further quadratic transformation at such a point removes the singularity completely. Hence the proper singularities of $F'$ can be removed. (3) Corresponding to the transformations which remove the proper singularities of $F'$ there are transformations removing the proper singularities of $F$.

*R. J. Walker* (Ithaca, N. Y.).

**Scott, D. B. Point-curve correspondences. III. Correspondences on a single surface.** Proc. Cambridge Philos. Soc. **45**, 342–353 (1949).

[The previous papers in this series appeared in the same Proc. **41**, 135–145 (1945); **42**, 229–239 (1946); these Rev. **7**, 27; **8**, 86.] The chief interest of the present paper lies in the definition of a point-curve correspondence with (nonzero) valency on a surface. Such a correspondence, by definition, induces on every proper curve on the surface a correspondence of double valency in which every invariant cycle is transformed into a cycle homologous to a nonzero multiple of itself while every vanishing cycle is transformed into a cycle which bounds on the curve. The author proves that every correspondence on a surface is a valency correspondence if, and only if, the Riemann matrix corresponding to the simple integrals of the first kind on the surface does not admit complex multiplication. However, unlike the somewhat analogous valency correspondences on a curve not every surface admits correspondences with nonzero valency; for such a correspondence to exist the algebraic intersection group of the surface must be cyclic, and conversely. An expression is obtained for the united curve of a point-curve correspondence with valency which is formally similar to the Chasles-Brill formula for the number of united points of a valency correspondence on a curve and this leads the author to consider various possible definitions of the valency of a point-curve correspondence with valency. He also outlines an extension of the ideas to point-curve correspondences with multiple valency. *J. A. Todd.*

**Tibiletti, Cesarina. Sulle curve multiple risolubili prive di punti di diramazione.** Ann. Mat. Pura Appl. (4) **27**, 87–99 (1948).

Le problème traité par l'auteur est de construire sur une courbe $f(x, y)$ de genre $p>1$, toutes les fonctions $z(x, y)$ birationnellement distinctes définies sur $f$, sans points de diramation solutions d'une équation de degré premier $n$, résoluble par radicaux. L'auteur établit topologiquement que si l'on se donne une courbe $f(x, y)$ et une fonction $v=\{g(x, y)\}^{1/k}$ ($k$ premier), à groupe cyclique, sans diramations, et irréductible, on peut choisir pour Riemannienne de $f$, une forme canonique telle que $v$ ne dirame que sur une seule rétrosection. Cette propriété topologique permet de ramener la construction cherchée à un cas particulier où la Riemannienne $R_f$ est un rectangle à trous rectangulaires. On construit alors simplement une fonction auxiliaire $w$, à groupe métacyclique (ou à un de ses sous-groupes) sans diramation; de cette fonction $w$ à $n(n-1)$ valeurs, on déduit facilement la fonction rationnelle $z$ par un emploi convenable de la génération du groupe métacyclique à partir de deux substitutions. Les constructions à effectuer se réduisent à des extractions de racines sur des Riemanniennes suivant les procédés jadis indiqués par Chisini [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **24**$_2$, 153–158 (1915)]. Le nombre des fonctions $z$ birationnellement distinctes peut s'obtenir simplement par le produit du nombre des fonctions, intermédiaires construites, birationnellement distinctes. *B. d'Orgeval* (Grenoble).

**d'Orgeval, Bernard. A propos des plans doubles de genres $p_a=p_g=0$.** C. R. Acad. Sci. Paris **228**, 975–976 (1949).

Campedelli [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **15**, 203–208, 358–362, 536–542 (1932)] a déterminé tous les plans doubles avec courbes de diramation du huitième ou du dixième ordre. Il y en a aussi de non rationels de genres arithmétique $p_a$ et géometrique $p_g$ tous les deux nuls. Ceux-ci ont tous un tacnode ou un point double uniplanaire auquel est infiniment voisin un tacnode. Si l'on regarde un de ces points singuliers comme virtuellement inexistant, on obtient un plan double de genres $p_a=p_g=1$, avec système bicanonique hyperelliptique. L'auteur recherche, inversement, tous les plans doubles non rationnels de genres $p_a=p_g=0$, qu'on obtient si l'on impose, aux plans doubles de genres $p_g=p_a=1$ avec système bicanonique hyperelliptique, un tacnode ou un point double uniplanaire auquel est infiniment voisin un tacnode, et trouve que, au dehors des plans doubles de Campedelli, sont possibles seulement deux autres cas: (1) un plan double de genre linéaire $p^{(1)}=3$, bigenre $P_2=3$ et courbe de diramation du douzième ordre avec un point sextuple, six points (3, 3) et un point quadruple; (2) un plan double avec $p^{(1)}=P_2=4$, et courbe de diramation du douzième ordre avec un point sextuple et sept points (3, 3). On appelle point (3, 3) un point triple auquel est infiniment voisin un autre point triple. On ne sait pas si ces deux plans doubles sont effectivement réalisables. *G. Zappa* (Naples).

**Stubban, John Olav. Sur la courbe d'intersection dans le cas où elle est dégénérée, entre $r-1$ hypersurfaces algébriques dans l'espace à $r$ dimensions.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 188–190 (1949).

Le but de cette note est de démontrer, par des méthodes géométrique-algébriques, la formule suivante:

$$(m_1-m_2)\{\sum(M_i-1)-2\}=2(\pi_1-\pi_2),$$

où les lettres $M_i$ signifient les ordres de $r-1$ hypersurfaces algébriques, dépourvues de points doubles, dans un espace à $r$ dimensions; on suppose que la courbe d'intersection de

ces hypersurfaces se décompose en deux parties d'ordres $m_1, m_2$ et de genres $\pi_1, \pi_2$ et dépourvues de points doubles. Il y a plusieurs fautes typographiques qui rendent la lecture difficile; par exemple, dans la formule (1) de p. 189 il faudrait écrire $s^{(h-1)}$ au lieu de $s^{(h+1)}$; on a $s^{(h)}_{i_1 i_2 \cdots i_h} = \tau_1$ pour $h = r-1$ et non pour $h=2$; etc. *E. G. Togliatti* (Gênes).

**Stubban, John Olav. Sur la courbe paracanonique.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 191–196 (1949).

On appelle courbe paracanonique une courbe algébrique $C^{2p-2}$, d'ordre $2p-2$ et de genre $p$, dans un espace $S_{p-2}$, qui représente une série linéaire paracanonique $g^{p-2}_{2p-2}$ sur une courbe de genre $p$. L'auteur s'occupe ici des courbes paracanoniques des genres 7, 6, 5; et en donne différentes constructions, en se servant aussi de la formula démontrée dans le travail précédent. Pour $p=7$ on a une $C_7^{12}$ de $S_5$ qui fait partie de l'intersection d'une forme cubique et de trois quadriques linéairement indépendantes; l'intersection complémentaire est aussi paracanonique. Pour $p=6$ on peut obtenir une $C_6^{10}$ paracanonique de $S_4$ comme intersection complémentaire de trois formes cubiques linéairement indépendantes contenant une même courbe $C_{20}^{17}$. Enfin, pour $p=5$, on a une $C_5^8$ de l'espace ordinaire qui fait partie de l'intersection de deux surfaces du quatrième ordre; l'intersection complémentaire est aussi paracanonique; on peut obtenir aussi des $C_5^8$ paracanoniques en coupant une surface cubique (réglée ou non) ou une quadrique par des surfaces convenables du quatrième ou du sixième ordre.

*E. G. Togliatti* (Gênes).

**Gaeta, Federico. Sulle curve sghembe algebriche di residuale finito.** Ann. Mat. Pura Appl. (4) **27**, 177–241 (1948).

The residual $\rho$ of an irreducible nonsingular curve $C$ in $S_r$ is the least integer such that there exists a set of irreducible nonsingular curves $C_0, C_1, \cdots, C_\rho = C$ such that each of $C_0, C_0+C_1, \cdots, C_{\rho-1}+C_\rho$ is the complete simple intersection of $r-2$ forms. The present paper develops an elaborate theory of such curves when $r=3$. The base for the ideal generated by surfaces passing through a curve of residual $\rho$ consists of exactly $\rho+2$ surfaces. Any curve of finite residual is arithmetically normal in the sense of Zariski, and the author shows that the converse is true, at any rate for an extended class of curves. There are numerous other results belonging to the same order of ideas, and some extensions to varieties of higher dimension. It is impossible to summarise the varied contents of this paper in a brief note.

*J. A. Todd* (Cambridge, England).

**Lage Sundet, Knut. On the reducibility of certain systems of plane algebraic curves.** Mat. Tidsskr. B. **1949**, 18–24 (1949).

L'auteur étudie les systèmes linéaires de courbes algébriques de genre virtuel $p$ et de dimension virtuelle $q$ qui vérifient $3p \leqq 2+q$. Il établit par une succession de transformations birationnelles une condition suffisante de réductibilité. Si un tel système est réductible il contient au moins une composante fixe rationnelle. En particulier une courbe décomposée virtuellement elliptique contient au moins une composante rationnelle. Une application est donnée comme condition nécessaire et suffisante d'existence d'une transformation Crémonienne plane ayant des points fondamentaux donnés. *L. Gauthier* (Nancy).

**Andreotti, Aldo. Sulle superficie di Kummer e di Weddle.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 61–64 (1949).

In his book [Serie, Sistemi d'Equivalenza e Corrispondenze Algebriche sulle Varietà Algebriche, v. 1, Edizioni Cremonese, Rome, 1942, p. 353; these Rev. **10**, 206] Severi has described the invariantive construction, from a curve $C$ of genus 2, of (i) a Jacobian surface $J$, image of point-pairs of $C$, and (ii) a Kummer surface $K$, image of the involution of pairs of points of $J$ which correspond to pairs of point-pairs of $C$ which are mutually residual in a $g_4^2$ on $C$. In the present paper, by similar invariantive methods, the author gives a simple and suggestive proof of the fact that the Weddle surface $W$ is birationally equivalent to a Kummer surface $K$, and that there exists a space Cremona transformation which carries $W$ into $K$.

The method used consists first in representing $C$ on a double twisted cubic $\Gamma$ with 6 branch points, and in showing that the $\infty^2$ $g_3^1$ on this double curve are generated by the tangent planes of the $\infty^2$ quadric cones through the branch points. The vertices of these cones generate a Weddle surface $W$, and the points of this surface are thus in birational correspondence with pairs of associated $g_3^1$ on $C$. The birational equivalence of $W$ with $K$ is then derived by associating any point-pair of $C$ (point of $J$) with the $g_3^1$ containing the triad formed by adding to the point-pair a fixed double point of the canonical $g_3^1$ on $C$. The author describes in detail the representation on $W$ of the nodes and tropes of $K$, and he determines the linear system of curves on $K$ of which $W$ is the projective model.

From the limiting case in which two of the double points of the $g_3^1$ on $C$ coincide, the author derives Plücker's quasi-hyperelliptic surface with a double line and extends to this his previous results. Finally he remarks that his results may be extended in part by considering the manifolds $W_{3r}$ in $S_{3r}$ which are loci of vertices of quadric cones whose tangent primes cut $g_{3r}^r$ on a double rational normal $\Gamma^{3r}$ of genus $2r$, the birational transforms of the $W_{3r}$ so arising being Wirtinger manifolds (models of involutions of order 2 on Jacobian manifolds relative to curves of genus $2r$).

*J. G. Semple* (London).

**Vaccaro, Giuseppe. Sopra alcune singolarità di ipersuperficie algebriche.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 247–251 (1948).

The author discusses what we may call inflexional $(n-2)$-fold points ($n \geqq 3$) of an irreducible primal $V^n_{r-1}$ of $S_r$, any such point $O$ having the property that every tangent ($n=3$) or nodal tangent ($n>3$) at $O$ has $n$ intersections with $V^n_{r-1}$ at $O$. He finds that a necessary and sufficient condition for an $(n-2)$-fold $O$ of $V^n_{r-1}$ to be inflexional is that $V^n_{r-1}$ should be invariant under a suitably chosen harmonic perspective with $O$ as vertex. If this is so, and if $\bar{\omega}$ is the axial space of the perspective, then any isolated double point $P$, distinct from $O$, which $V^n_{r-1}$ may possess must lie in $\bar{\omega}$. The author discusses the cases in which the line $OP$ does or does not lie on $V^n_{r-1}$. He shows that in the first case $V^n_{r-1}$ has a fixed tangent prime along $OP$. For $n=4$, the maximum number (always attainable) of inflexional double points which the $V^4_{r-1}$ can possess is $r+1$; also, if it has $r+1$ double points of which all but one are inflexional, then so also is the last.

*J. G. Semple* (London).

**Hartley, E. M. Some determinantal quartic primals in four dimensions.** Proc. Cambridge Philos. Soc. **45**, 43–49 (1949).

Dans un espace $S_3$ à trois dimensions, considerons dix droites $a_{ik}$ ($i=1, \cdots, 5$; $k=1, 2$); $a_{ik}$ est incident à $a_{jh}$ ($k=h$ ou $k\neq h$) seulement si $j-i\equiv\pm1 \pmod 5$. Ces droites forment une courbe $C$ d'ordre 10 et genre virtuel 11. L'image projective, dans un espace $S_4$ à quatre dimensions, du système linéaire $\Sigma$ des surfaces du quatrième ordre qui passent par $C$, est une hypersurface $F$ du quatrième ordre, considerée par J. A. Todd [Quart. J. Math., Oxford Ser. **6**, 129–136 (1935)]. La $F$ est un cas particulier de l'hypersurface déterminantale du quatrième ordre [T. G. Room, The Geometry of Determinantal Loci, Cambridge University Press, 1938], et a, ordinairement, 40 noeuds; pour des $C$ plus particulières on obtient une hypersurface $F$ avec 45 noeuds [Todd, Quart. J. Math., Oxford Ser. **7**, 168–174 (1936)]. L'auteur, dans ce travail, montre que, en faisant coïncider entre eux quelques-unes des 20 intersections des droites $a_{ik}$, on obtient, de suite, des hypersurfaces avec 41, 42, 43, 44, 45 noeuds, et étude, pour chaque hypersurface, la configuration formée des nodes et des plans qui se trouvent sur elle. *G. Zappa* (Naples).

**Conforto, Fabio. Sopra le corrispondenze univoche tra i punti di una varietà quasi abeliana di Picard, rappresentate da congruenze lineari tra gli integrali virtualmente di prima specie.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 369–375 (1948).

Il lavoro si svolge nell'ambito della teoria delle funzioni quasi-abeliane di $\pi$ variabili $u_1, \cdots, u_\pi$, recentemente costruita da F. Severi [Funzioni quasi-abeliane, Pontificiae Acad. Scientiarum Scripta Varia, v. 4, 1947; queste Rev. **9**, 578]. Si studiano, sulla varietà di Picard $V_\pi$ associata ad un corpo di funzioni quasi-abeliane, le corrispondenze univoche rappresentabili mediante congruenze lineari tra matrici del tipo $u' \equiv \Lambda u+\lambda \pmod\omega$, ove $u$, $u'$ sono le matrici che compendiano le $\pi$ variabili relative ai due punti corrispondenti, ed $\omega$ è la matrice dei periodi formata da $\pi$ righe e $\pi'<2\pi$ colonne. L'autore dimostra, fra l'altro, che le corrispondenze non degeneri risultano, come nel caso abeliano, di indici $\alpha$, 1, ma accade qui il fatto inaspettato e nuovo che tali corrispondenze non sono necessariamente algebriche.
*E. Martinelli* (Genova).

**Godeaux, Lucien. Sur les variétés de Segre généralisées.** Bull. Soc. Roy. Sci. Liége **18**, 20–24 (1949).

The reviewer introduced the manifold representative of the points of $k$ linear spaces $S^j$, $j=1, \cdots, k$, of dimension $r$; taken with multiplicities $n_j$ (topological product of multiple spaces). The author proves that the same manifolds (which he had already studied in particular cases) are obtained by considering algebraic equations of order $n_j$ in the coordinates of $S^j$ and by representing these equations on the hyperplanes of a convenient hyperspace. Their order, already known, is found by considering the manifold as belonging to a Segre manifold relative to spaces $\Sigma^j$ of dimensionality $(n_{j,r_j}+r_j)-1$. Numerical examples are given.
*E. Bompiani* (Rome).

**Godeaux, Lucien. Sur les points de diramation isolés des surfaces multiples. I.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 15–30 (1949).

La recherche contenue dans cette note se rattache aux nombreux travaux de l'auteur sur les involutions cycliques appartenant à une surface algébrique et n'ayant qu'un nombre fini de points unis. L'auteur considère ici la surface multiple $\Phi$ qui représente, dans un espace à un nombre convenable de dimensions, une involution $I_p$ d'ordre premier $p=2\nu+1$ existant sur une surface $F$; à un point uni $A$ isolé de $I_p$ correspond sur $\Phi$ un point de diramation isolé $A'$. L'auteur se propose ici d'étudier la structure du point $A'$ sur $\Phi$; $A'$ a pour $\Phi$ une certaine multiplicité; et le cône tangent à $\Phi$ en $A$ se décompose en deux ou en plus que deux cones. Dans la note présente, en poursuivant des recherches antérieures, l'auteur considère le cas où ce cône se décompose en deux parties seulement. L'instrument principal de la recherche est constitué par un système linéaire $|C|$ de courbes de $F$ contenant $p$ systèmes linéaires $|C_0|, |C_1|, \cdots, |C_{p-1}|$ appartenant à l'involution $I_p$; aux courbes de $|C|$ correspondent sur $\Phi$ les sections hyperplanes.
*E. G. Togliatti* (Gênes).

**Godeaux, Lucien. Sur une représentation des transformations birationnelles de l'espace.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 92–96 (1949).

Une transformation birationnelle $T$ entre deux espaces $\Sigma$ et $\Sigma'$ projectifs à trois dimensions, associe respectivement les plans $a$ et les surfaces $A$ de $\Sigma$ aux surfaces $A'$ et aux plans $a'$ de $\Sigma'$. On rapporte projectivement le système complet $|a+A|$ aux hyperplans d'un espace $S_r$: aux points de $\Sigma$ correspondent les points d'une $V_3$ de $S_r$ qui représente aussi les points correspondants de $\Sigma'$. Dans cette image $V_3$ des couples de $T$, les plans $a$ correspondent à un réseau homaloïdal $|F|$ de surfaces rationnelles d'ordre $2n+n'+1$. Les éléments fondamentaux de première espèce sont représentés sur $V_3$ par des surfaces rationnelles. Soit $\Gamma$ une courbe fondamentale de seconde espèce d'ordre $\nu$ et multiple d'ordre $s$ pour $A$ ($\Gamma\to\Gamma', \nu', s'$). On a $s\nu=s'\nu'$. Un point $P$ de $\Gamma$ est représenté sur $V_3$ par une courbe $\gamma$ rationnelle d'ordre $s$ qui décrit, quand $P$ parcourt $\Gamma$, une surface $M$ sur laquelle les $\gamma$ forment un faisceau. L'étude de cette surface $M$ permet d'approfondir l'étude des courbes fondamentales de seconde espèce. *L. Gauthier* (Nancy).

**Godeaux, Lucien. Sur l'existence de certaines surfaces doubles.** Bull. Soc. Roy. Sci. Liége **16**, 128–133 (1947).

**Godeaux, Lucien. Observations sur la structure des points de diramation des surfaces multiples cycliques.** Bull. Soc. Roy. Sci. Liége **16**, 242–247 (1947).

**Godeaux, Lucien. Sur la construction de modèles de surfaces algébriques contenant des involutions cycliques.** Bull. Soc. Roy. Sci. Liége **17**, 56–61 (1948).

**Godeaux, Lucien. Sur la structure des points unis isolés des involutions cycliques appartenant à une surface algébrique.** Bull. Soc. Roy. Sci. Liége **17**, 116–127 (1948).

**Godeaux, Lucien. Sur la structure des points unis des involutions cycliques appartenant à une surface algébrique. Étude d'un exemple.** Bull. Soc. Roy. Sci. Liége **17**, 127–137 (1948).

**Godeaux, Lucien. Sur les surfaces irrégulières contenant un système linéaire isolé.** Bull. Soc. Roy. Sci. Liége **17**, 196–199 (1948).

**Godeaux, Lucien. Remarques sur les points unis isolés des involutions cycliques appartenant à une surface algébrique.** Bull. Soc. Roy. Sci. Liége **17**, 297–303 (1948).

**Godeaux, Lucien. Sur les transformations de Jonquières involatives.** Bull. Soc. Roy. Sci. Liége **18**, 9–19 (1949).

**Godeaux, Lucien. Exemple de la détermination de la singularité d'une surface multiple en un point de diramation isolé.** Bull. Soc. Roy. Sci. Liége **18**, 38–45 (1949).

**Derwidué, L. Sur la génération des transformations birationnelles de l'espace.** Bull. Soc. Roy. Sci. Liége **16**, 134–144 (1947).

**Jongmans, F. A propos d'une propriété des surfaces polaires.** Bull. Soc. Roy. Sci. Liége **16**, 289–291 (1947).

**Lorent, H. Sur une famille de sécantes d'une cubique plane de genre un.** Bull. Soc. Roy. Sci. Liége **17**, 143–149 (1948).

**Lorent, H. Surfaces algébriquement associées aux quadriques réglées.** Bull. Soc. Roy. Sci. Liége **17**, 245–252 (1948).

**Hans-Frère, Andrée. Sur la transformation birationnelle du troisième ordre de l'espace.** Bull. Soc. Roy. Sci. Liége **17**, 150–154 (1948).

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## *Differential Geometry*

**Behari, Ram. A note on geodesic curvature.** Math. Student **15** (1947), 100–101 (1949).

L'auteur indique un procédé rapide pour retrouver l'expression de la courbure géodésique des trajectoires orthogonales des génératrices d'une surface réglée. Si $P$ décrit l'une de ces trajectoires orthogonales, il existe sur la génératrice issue de $P$ un point $M$ et un seul dont la vitesse est constamment perpendiculaire à celle de $P$, et ce point, de détermination analytique aisée, est le centre de courbure géodésique en $P$ de la trajectoire orthogonale envisagée.

*P. Vincensini* (Marseille).

**Ascoli, Guido. Sopra une proprietà delle normali ad una superficie ed una sua parziale estensione. II.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 3–6 (1949).

Part II treats the exceptional cases of part I [same Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 280–285 (1948); these Rev. **10**, 570]. For a surface which is not isometric to a surface of revolution the only congruence with property $E$ is that of the normals. For a surface isometric to a surface of revolution, but not with constant curvature, there are $\infty^1$ congruences with property $E$ which are obtained from each other by refraction in the surface (according to the optical law). For surfaces of constant curvature there are $\infty^3$ congruences with property $E$. Surfaces with curvature 0 are the only ones for which congruences formed by tangents can have property $E$. *H. Busemann* (Los Angeles, Calif.).

**Petrovitch, Michel. Courbes planes dont la courbure est fonction monotone de la longueur de l'arc.** Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. **7**, 61–69 (1941).

L'auteur étudie le problème de la détermination d'une courbe plane dont la courbure est une fonction continue et monotone $f(s)$ de l'arc $s$ compté à partir d'un point fixe de la courbe. Il envisage quatre cas, différant par le mode de variation de $f(s)$, et indique, pour chacun d'eux, la forme (spirale) de la courbe, la région du plan qu'elle occupe, ses éléments asymptotiques (point, cercles, ou droite), et ses tangentes de direction donnée. Une étude spéciale est faite du cas où la courbure est proportionnelle à l'arc, avec développement des coordonnées cartésiennes $(x, y)$ d'un point quelconque de la courbe, et application de ces développements à la détermination d'approximations algébriques pour des arcs donnés de la courbe envisagée.

*P. Vincensini* (Marseille).

**Keller, Ott-Heinrich. Zur Theorie der ebenen Berührungstransformationen. I. Fundamentalkurven ebener Berührungstransformationen.** Math. Ann. **120**, 650–675 (1949).

A study of contact-transformations

$$\begin{aligned} X&=X(x,y,z), & x&=x(X,Y,Z),\\ Y&=Y(x,y,z), & y&=y(X,Y,Z),\\ Z&=Z(x,y,z), & z&=z(X,Y,Z) \end{aligned}$$

($dx=z\,dy$, $dX=Z\,dY$) in the neighborhood of fundamental elements, i.e., of elements $(x, y, z)$ for which the functions $(X, Y, Z)$ become indeterminate. It is supposed that for such elements the corresponding functions are quotients of integral power series of the coordinates of the element and that the singular points (centers of those elements) have a generic position on a curve (isolated singular points are not taken into consideration): a fundamental curve. There are two kinds of fundamental curves. In the first kind, the image of an element depends on the branch of the curve on which we approach the given element (on the order of contact of the branch with the given element). In the second kind the image of an element depends on the way in which a variable element approaches a given element, its center moving in a direction different from that of the given element. Fundamental curves of the first kind are also, in general, fundamental curves of the second type.

These singularities already appear in algebraic birational contact transformations [studied by L. Antonne, J. Math. Pures Appl. (4) **4**, 177–247, 407–464 (1888); and G. Fano, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **8**, 445–451 (1928)]; and, according to the author, it is always possible to obtain a given singularity by Cremona transformations and collineations (which are contact transformations). [The legitimacy of the procedures is not always fully justified (e.g., the use of Bertini's theorem for a system of nonalgebraic curves; the noninvariance of some definitions).] *E. Bompiani* (Rome).

**Segre, Beniamino. Generalizzazione di un teorema di Beltrami.** Boll. Un. Mat. Ital. (3) **4**, 16–22 (1949).

The main theorem, which includes as special cases two classical theorems of Beltrami and of Busse, is this: the only surfaces which can be put into correspondence with the Euclidean plane so that geodesics map into circles (or straight lines) are of constant curvature. The proof is straightforward. Later in the paper a more synthetic proof is given. Furthermore it is shown that for any given surface

of constant curvature the above mappings depend on eleven parameters and are compounded of a surface to plane geodesic correspondence and a quadratic fractional transformation. Further study of this last transformation and extensions to higher dimensions are reserved for a later paper. *J. L. Vanderslice* (College Park, Md.).

**Yaglom, I. M. Operational methods in differential geometry.** Uspehi Matem. Nauk (N.S.) **4**, no. 1(29), 194–204 (1949). (Russian)

As a substitute for the classical theory of surfaces based on quadratic differential forms a theory based on differential operators is proposed. Trihedra consisting of three mutually perpendicular unit vectors are placed on the surface in all possible ways, such that one of the vectors coincides with the normal unit vector of the surface. The infinitesimal operators used are those under which this set of trihedra is invariant. Two choices of basic operators are used, each consisting of three operators. The Beltrami and other invariant operators are expressed (quadratically) in terms of the basic operators. The coefficients appearing in the commutation relations among the basic operators are shown to have geometrical meaning in terms of normal curvature and geodesic torsion, and the usual invariants are written in terms of them. The counterparts of the Gauss and Codazzi equations are given. The surface is considered as imbedded in a space of constant curvature rather than in Euclidean space, since this generalization does not result in complications. *G. Y. Rainich* (Ann Arbor, Mich.).

**Pa, Chenkuo. Some theorems on rectilinear congruences and transformations of surfaces.** Trans. Amer. Math. Soc. **65**, 360–371 (1949).

L'auteur étudie la configuration géometrique constituée par une surface $S$ et une congruence rectiligne $L$ dont les rayons sont issus des différents points de $S$ et non situés dans les plans tangents correspondants. Les principaux résultats obtenus sont formulés par les deux théorèmes suivants. (I) Si $L$ est $W$ et si les asymptotiques se correspondent sur $S$ et sur les réseaux focaux de $L$, $S$ est une surface $R$, et $L$ une congruence $R$ dont les développables déterminent un réseau $R$ sur $S$; en outre, à toute surface $R$ sont associées $\infty^4$ congruences $L$ du type $R$ jouissant des propriétés indiquées. (II) Si $L$ est conjuguée à $S$, et si $S'$ est la surface lieu du conjugué harmonique, par rapport aux foyers d'un rayon quelconque de $L$, du point de $S$ d'où ce rayon est issu, la correspondance des asymptotiques sur $S$ et $S'$ exige que $S$ et $S'$ soient deux surfaces de Jonas sur lesquelles les développables de $L$ déterminent deux réseaux de Jonas; à toute surface de Jonas on peut d'ailleurs associer $\infty^3$ congruences $L$ donnant lieu à la configuration indiquée. Au théorème (I) se trouve rattachée la recherche des surfaces $S$ dont les normales projectives forment une congruence $W$ sur les nappes focales de laquelle les lignes asymptotiques correspondent à celles de $S$. Ces surfaces sont des surfaces $R$ spéciales définies par des relations caractéristiques déterminées. Une autre classe de surfaces $R$ s'obtient en exprimant que la congruence $W$ qui intervient dans le théorème (I) est formée par les droites canoniques $C(\lambda)$ $(\lambda \neq 0)$, du premier faisceau canonique de $S$. Les surfaces $S$ auxquelles on est ainsi conduit sont des surfaces $R$ isothermes asymptotiques généralisant celles pour lesquelles la droite canonique $C(\lambda)$ passe par un point fixe. Si l'on exige que la congruence $L'$, réciproque par rapport à $S$ de la congruence $L$ du théorème (I), soit $W$ en même temps que $L$, $S$ doit appartenir à une nouvelle classe de surfaces dont sont données les relations caractéristiques de définition. Au sujet du théorème (II), l'auteur signale le cas où $S'$ est un plan. La correspondance des asymptotiques entre $S$ et $S'$ est alors réalisée, et $S$ est une surface de Jonas sur laquelle le réseau de Jonas est déterminé par les développables de $L$. Il discute ensuite en détail le cas spécial où les développables de $L$ sont indéterminées, et où par suite $S$ et $S'$ sont mises par $L$ en correspondance perspective. Il est, en particulier, amené à généraliser le théorème de Koenigs pour les réseaux plans asymptotiques, et à montrer que si deux surfaces $S$ et $S'$ sont en correspondance perspective, et si les asymptotiques de $S$ correspondent à un réseau conjugué $\Omega$ de $S'$, $\Omega$ est à invariants égaux, tout réseau à invariants égaux étant d'ailleurs perspectif d'un réseau asymptotique. Le mémoire se termine par l'étude des conditions pour qu'une congruence conjuguée à une surface donnée $S$ (à un réseau conjugué de $S$) soit $W$.
*P. Vincensini* (Marseille).

**Marcus, F. Sur la définition des couples de congruences de droites stratifiables.** Bull. Soc. Roy. Sci. Liége **18**, 96–99 (1949).

A pair of congruences $K$, $K'$ are said to be stratifiable in one sense if they may be put in one-to-one line correspondence such that the planes through the lines of $K'$ envelope a one parameter of surfaces $S$, the characteristic points of the surfaces $S$ lying on the corresponding lines of $K$. They are stratifiable in both senses, if moreover, there exist a family of surfaces $S'$ whose tangent planes at their intersection with the lines of $K'$ pass through the corresponding lines of $K$. The purpose of this note is to prove the following theorem. A pair of congruences $K$, $K'$ stratifiable in one sense is stratifiable in both senses if there exist two surfaces $S'$ (not a one-parameter family) whose tangent planes at their points of intersection with the lines of $K'$ pass through the corresponding lines of $K$. [The reviewer, Ann. of Math. (2) **43**, 623–633 (1942); these Rev. **4**, 53, has shown that a pair of congruences $K$ and $K'$ are stratifiable in both senses (are in relation $T$) if there exist three transversal surfaces of each congruence $K$, $K'$ whose tangent planes at their intersections with the lines of one congruence pass through the corresponding lines of the other.]
*V. G. Grove* (East Lansing, Mich.).

**Rozet, O. Sur certaines congruences non $W$ de droites.** Bull. Soc. Roy. Sci. Liége **17**, 6–9 (1948).

A congruence which is not $W$ has $S$ as one of its focal surfaces. Analytic conditions that the other focal surface be a nondevelopable ruled surface (and, in particular, a quadric) are found. *V. G. Grove* (East Lansing, Mich.).

**Rozet, O. Sur certaines classes de congruences non $W$ de droites.** Bull. Soc. Roy. Sci. Liége **17**, 186–188 (1948).

The images $U$, $V$ in $S_8$ of the tangent lines to the asymptotic curves of a surface $S$ generate a Laplace sequence of conjugate nets $\cdots, U_2, U_1, U, V, V_1, V_2, \cdots$. Congruences $(g)$, which are not $W$ and have $S$ as one focal surface, correspond in $S_8$ to a surface $(G)$, the locus of the image point $G$ of the line $g$ of $(g)$. Congruences $(g)$ are found whose image points $G$ generate a net on $(G)$ whose Laplace transforms $G_{-1}$, $G_1$ of the second kind coincide with $U_2$ or $V_2$. In particular, a congruence is found having $S$ as one focal surface, the other being the locus of the intersections of the lines of $(g)$ with the second directrix of Wilczynski, the

focal planes being the tangent plane to $S$ and the plane determined by the line of $(g)$ and the first directrix of Wilczynski. The transforms $G_1$ and $G_{-1}$ coincide with $V_2$ and $U_2$, respectively. *V. G. Grove* (East Lansing, Mich.).

**Rozet, O. Sur les complexes satellites des congruences de droites.** Bull. Soc. Roy. Sci. Liége **17**, 189–193 (1948).

The satellite complex $\Gamma$ of a congruence $(g)$ relative to a focal point $x$ is the linear complex containing the lines through the other focal point $\bar{x}$ and lying in the tangent plane to the locus of $x$ at $\bar{x}$ and tangent to $(g)$ along $g$. If $(g)$ is a $W$-congruence then $\Gamma$ coincides with the osculating linear complex of $(g)$ along $g$, and depends on one parameter. If $(g)$ is not a $W$-congruence, $\Gamma$ depends on two parameters. *V. G. Grove* (East Lansing, Mich.).

**Rozet, O. Sur une surface dont la transformée de Lie est la surface minima d'Enneper.** Bull. Soc. Roy. Sci. Liége **17**, 208–209 (1948).

**Rozet, O., et Thibaut, F. Sur les surfaces dont les cyclides de Lie n'ont que deux points caractéristiques.** Bull. Soc. Roy. Sci. Liége **17**, 210–214 (1948).

**Backes, F. Sur une classe de transformations de Darboux.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 50–59 (1949).

Darboux a découvert et étudié les congruences de sphères dont les deux nappes sont en correspondance isotherme conforme. L'auteur du présent article applique la méthode du pentasphère mobile de A. Demoulin à la recherche des congruences de Darboux pour lesquelles la sphère génératrice coupe une sphère fixe sous un angle constant. Il montre que les congruences de Darboux jouissant de cette propriété sont en relation avec les surfaces à courbure totale constante. A toute surface à courbure totale constante correspond une congruence de Darboux dont les sphères coupent une sphère fixe sous un angle constant; en outre, à toute congruence de Darboux jouissant de la propriété indiquée, on peut associer une infinité d'autres congruences de Darboux jouissant de la même propriété. Soit $\Gamma$ le cercle ayant pour foyers les deux points où la sphère $S_1$ génératrice de l'une quelconque des congruences étudiées touche son enveloppe; ce cercle engendre un système cyclique et est cosphérique avec le cercle intersection de $S_1$ et de la sphère fixe $S_f$ que $S_1$ coupe à angle constant; l'auteur envisage la congruence engendrée par la sphère $\Sigma$ contenant les deux cercles $\Gamma$ et $(S_1, S_f)$; cette congruence est une congruence de Ribaucour, et le calcul de son élément angulaire montre qu'elle est à courbure constante. Si l'on suppose la courbure égale à un, on retrouve un cas connu, caractérisé par le fait que le cercle ayant pour foyers les points caractéristiques de $\Sigma$ engendre un système cyclique. *P. Vincensini.*

**Blank, Ya. P. Conjugate nets of conical lines.** Doklady Akad. Nauk SSSR (N.S.) **64**, 755–758 (1949). (Russian)

A curve on a surface is called conical if $(\xi d\xi d\xi^2 d\xi^3) = 0$; a conjugate net is said to be a Peterson net if its curves are conical. If the family of Peterson nets depends on four parameters the surface is a quadric. There are no surfaces possessing $\infty^3$ Peterson nets. A surface possessing $\infty^2$ Peterson nets can be brought by a projective transformation into one of the forms

$$z=(y/x)^n, \quad z=\log(y/x), \quad z=y/x+x^3, \quad z=y/x+\log x.$$

*G. Y. Rainich* (Ann Arbor, Mich.).

**Lalan, Victor. Sur la déformation à réseau conjugué persistant.** C. R. Acad. Sci. Paris **228**, 1842–1844 (1949).

The problem considered is the determination of those surfaces which can be deformed isometrically (in the small) with preservation of the conjugate net (provided the curves of the conjugate net are not geodesics). Using the methods of É. Cartan the author reduces this problem to the solution of a system of partial differential equations consisting of a single equation of the fourth order in one unknown and a pair of equations of the first order in two unknowns. By specializing the arbitrary functions involved various known solutions are obtained. *C. B. Allendoerfer.*

**Lemoine, Simone. Le rôle des directions principales dans l'immersion des surfaces.** Bull. Sci. Math. (2) **72**, 168–190 (1948).

Using a new approach and the method of differential forms, the author discusses several related topics previously treated by É. Cartan [Les systèmes différentiels extérieurs et leurs applications géométriques, Actual. Sci. Ind., no. 994, Hermann, Paris, 1945; these Rev. **7**, 520]. Part I is devoted to the question of finding those surfaces which have a given $ds^2$ and principal directions. For a specific choice of these there are at most two such surfaces, except in special cases noted below. The class of surfaces for which a second surface of this type exists depends on four arbitrary functions of a single argument. In part II the author examines the restrictions upon possible choices of principal directions imposed by the choice of $ds^2$. It is shown that the principal directions are obtained by integrating a system of two third-order partial differential equations. The exceptional cases are treated in part III. First are the cylindrical molding surfaces for which the surfaces isometric to a given surface and with corresponding lines of curvature depend upon a single arbitrary constant. Finally there are the developable surfaces for which standard results are proved. *C. B. Allendoerfer* (Haverford, Pa.).

**Lemoine, Simone. Détermination des couples de surfaces isométriques avec correspondance des lignes de courbure.** C. R. Acad. Sci. Paris **228**, 1476–1478 (1949).

A new demonstration is given of the following theorem of É. Cartan [Les systèmes différentiels extérieurs et leurs applications géométriques, Actual. Sci. Ind., no. 994, Hermann, Paris, 1945, in particular, p. 148; these Rev. **7**, 520]. In general, isometric pairs of surfaces with corresponding lines of curvature form a class depending on four functions of a single argument. *C. B. Allendoerfer.*

**Bompiani, E. Sulle corrispondenze puntuali fra spazi proiettivi.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 145–151 (1949).

This note deals with certain projective correspondences subordinate to a general point transformation between two projective spaces. Only the second order effects of the latter in the neighborhood of a given point are considered. Third order effects will be studied in a later note. A main point of the paper is how some geometrical entities apparently essential in setting up the correspondences disappear from the final results. Here is a typical case. Given a point transformation between two planes and a pair of corresponding points $O$, $O'$, any $E_2$ (linear element of second order) having an assigned tangent at $O$ together with another given straight line through $O$ determine a projectivity between the points of the tangent at $O$ and the corresponding tangent

at $O'$ which is independent of the particular $E_2$ chosen. This is generalized to $n$ dimensions in several ways. One by-product: a transformation between two $n$-spaces given up to the second order at two corresponding points $O$, $O'$ plus a given ennuple of lines through $O$ determines a projectivity between the two spaces. *J. L. Vanderslice.*

**Strubecker, Karl. Über die parataktische Abbildung der Flächenelemente des isotropen Raumes auf Punktepaare einer Ebene.** J. Reine Angew. Math. **186**, 129–164 (1945).

This paper is related to publications of Fubini, Hjelmslev and Study on representation of straight lines of elliptic space by point pairs on a sphere. Isotropic space is the space with coordinates $x$, $y$, $z$ and the linear element $ds^2=dx^2+dy^2$. It is a limiting case of elliptic space. The group is the group with 6 parameters of the isotropic movements. For parallel points and parallel planes a distance can be defined. There are two commutative groups of translations called isotropic right- and left-translations and named after Clifford; further, left-complexes invariant for all right-translations and right-complexes invariant for all left-transformations. Planar elements that are transformed into each other by an isotropic left- or right-translation are called right- or left-paratactic. Parallel elements are defined as simultaneously right- and left-paratactic. For isotropic space the representation mentioned above meets with difficulties. Therefore the author takes planar elements instead of linear elements, an idea used already by Scheffers but in quite another case. Regular planar elements can be represented linearly by point pairs in an auxiliary ordinary plane. This is called the paratactical representation, The geometrical properties of this representation are first dealt with. Then full details are given on the paratactical representation of the fundamental isotropic invariants. The paratactical representations of linear manifolds of planar elements and of some groups of transformations in isotropic space are discussed. The last section deals with the relations between the paratactical representation and the theory of transformations in a plane that leave areas invariant. This long paper contains many details which cannot be mentioned in this review. *J. A. Schouten* (Epe).

**Walker, A. G. On Lichnerowicz's conjecture for harmonic 4-spaces.** J. London Math. Soc. **24**, 21–28 (1949).

Le rapporteur a émis [Bull. Soc. Math. France **72**, 146–168 (1944); ces Rev. **7**, 80] la conjecture que tout espace riemannien harmonique $H_4$ à métrique définie positive était symétrique au sens de Cartan. L'auteur démontre la vérité de cette conjecture. Tout $H_n$ de type hyperbolique normal étant à courbure constante [Lichnerowicz et Walker, C. R. Acad. Sci. Paris **221**, 394–396 (1945); ces Rev. **7**, 395], il ne reste plus, pour $n=4$, qu'à étudier les espaces de signature 0. A ce sujet, l'auteur établit qu'il existe des espaces riemanniens $V_4$, de signature 0, satisfaisant aux équations algébriques en $g_{ij}$, $R_{ijkl}$, $\nabla_m R_{ijkl}$ vérifiées par les $H_4$ et qui ne sont ni symétriques ni récurrents au sens de Ruse [J. London Math. Soc. **21** (1946), 243–247 (1947); ces Rev. **9**, 102]. *A. Lichnerowicz* (Strasbourg).

**Trabant, E. A. Some interesting Riemann spaces.** Revista Ci., Lima **50**, 171–180 (1948).

It is shown that necessary and sufficient conditions that a Riemann space whose first fundamental form is $ds^2=f(x_1)[dx_1^2+dx_2^2+\cdots+dx_n^2]$ be either an Einstein space or a space of constant curvature is that $f(x_1)=K^{-1}(x_1+c)^{-2}$. This theorem is contained in known results. [For example, see L. P. Eisenhart, Riemannian Geometry, Princeton University Press, 1926, p. 85, equation (27.3).] *A. Fialkow* (Brooklyn, N. Y.).

**Castoldi, L. Dislocazioni e deformazioni finite nelle varietà sostanziali di Riemann.** Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 373–392 (1948).

The author considers the finite deformation of a $V_m$ immersed in a $V_n$, in which the displacement takes place. Formulae are worked out for the dilatation of any subspace of the $V_m$, for the change in the intrinsic curvature of the $V_m$, and for the change in the relative curvature of the $V_m$ with respect to the $V_n$. *A. J. McConnell* (Dublin).

**Van Bergen, F. Mouvement d'un solide dans un espace riemannien.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 186–187 (1949).

A movement is solid if the velocity field satisfies the condition $V_{a,b}+V_{b,a}=0$ [De Donder, same Bull. Cl. Sci. (5) **28**, 8–16 (1942); these Rev. **7**, 176]. The author uses well-known identities for the Riemann curvature tensor to show that these equations lead to the equations $V_{a,b,c}=R^s_{c,ba}V_s$ when the fundamental tensor is general rather than constant as De Donder took it. *A. Schwartz* (New York, N. Y.).

**Egorov, I. P. On the groups of motion of spaces with asymmetric affine connection.** Doklady Akad. Nauk SSSR (N.S.) **64**, 621–624 (1949). (Russian)

Let $x^1, x^2, \cdots, x^n$ be the coordinates of a point in a space with affine connection in which the object $\Lambda^\alpha_{\beta\gamma}$ is not symmetrical, i.e., $\Lambda^\alpha_{(\beta\gamma)}\neq 0$. Let us denote by $D_\beta v^\alpha=\partial v^\alpha/\partial x^\beta-\Gamma^\alpha_{\beta\gamma}v^\gamma$ covariant differentiation formed with $\Gamma^\alpha_{\beta\gamma}=\Lambda^\alpha_{(\beta\gamma)}$. Let $\Lambda^\alpha_{[\beta\gamma]}=\Omega^\alpha_{\beta\gamma}$ be the torsion tensor and $R^\alpha_{\beta\gamma\delta}$ the curvature tensor in the space with the connection $\Gamma^\alpha_{\beta\gamma}$. Lie derivation with the tensor $\Omega^\alpha_{\beta\gamma}$ shall be denoted by $D_L$. The differential operator $XF=v^\alpha\partial F/\partial x^\alpha$, where $v^\alpha(x^1,\cdots,x^n)$ is a covariant vector, defines a motion in the space, when

$$(1)\qquad D_\beta v^\alpha=u_\beta{}^\alpha,\quad D_\gamma u_\beta{}^\alpha=R^\alpha_{\beta\gamma\sigma}v^\sigma,\quad D_L\Omega^\alpha_{\beta\gamma}=0.$$

The integrability conditions of equations (1) with respect to the unknown functions $v^\alpha$, $u_\beta{}^\alpha$ are

$$(2)\qquad D_L\Omega^\alpha_{\beta\gamma}=0,\quad D_LR^\alpha_{\beta\gamma\delta}=0,\quad D_L(D_\mu\Omega^\alpha_{\beta\gamma})=0,\cdots,$$
$$D_L(D_{\mu_1}D_{\mu_2}\cdots D_{\mu N-3}R^\alpha_{\beta\gamma\delta})=0,$$
$$D_L(D_{\mu_1}\cdots D_{\mu N-1}\Omega^\alpha_{\beta\gamma})=0.$$

A necessary and sufficient condition that there exists a complete group of order $r$ of motions (that is, a Lie group of $r$ linearly independent operators $XF$, which has the maximum order in the given space) is that there exists a positive integer $N$ such that the ranks of $N$ series and of $N+1$ series of integrability conditions with respect to $v^\alpha$, $u_\beta{}^\alpha$ are equal to $n^2+n-r$.

The author proves the following theorems. (1) The maximum order of complete group of motions in a space with asymmetric connection is $n^2$. (2) The spaces with symmetric affine connection in which there exists a complete group of motions of order $n^2$ are the projective-Euclidean spaces. [See also Doklady Akad. Nauk SSSR (N.S.) **61**, 605–608 (1948); these Rev. **10**, 211.] *F. Vyčichlo.*

**Gheorghiu, Octavian Em. Équations aux dérivées partielles et objets géométriques.** Bull. Sci. Tech. Polytech. Timişoara **13**, 223–233 (1948).

The general wave equation $\theta(f)\equiv b^{ij}\partial_i\partial_j f+c^i\partial_i f+af=0$, where $f$ and $\theta(f)$ are supposed to be densities of weight

$p$ and $q$, is written in an invariant form by means of an affine connection $\Gamma^{s}_{jt}$ [see N. Théodoresco, Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] 12, 9–16 (1941); these Rev. 7, 80]. *J. Haantjes* (Leiden).

**Nagabhushanam, K. An extension of a theorem of Wintner.** Proc. Nat. Inst. Sci. India 15, 35–36 (1949).

Consider an arbitrary alternating tensor $A_{i\ldots l}$ of order $m$ ($m<n$) in an affine space of $n$-dimensions. By use of a tensor $a_{k}{}^{j}$, one can form the alternating tensor $a_{[k}{}^{j}A_{i\ldots]j]}$. The author proves that if the arbitrary alternating tensor $A_{i\ldots l}$ is such that the vanishing of its Stokes tensor (generalized curl) implies the vanishing of the Stokes tensor of $a_{[k}{}^{j}A_{i\ldots]j}$ then $a_{k}{}^{j}=\mu\delta_{k}{}^{j}$, where $\mu$ is a constant and $\delta_{k}^{j}$ is the Kronecker delta. This result generalizes a theorem of A. Wintner [The Analytical Foundations of Celestial Mechanics, Princeton University Press, 1947; these Rev. 3, 215]. Finally, an alternative generalization is furnished by use of epsilon-conjugate tensors. *N. Coburn* (Ann Arbor, Mich.).

**Moreau, Jean-Jacques. Le calcul tensoriel et les opérations vectorielles orientées.** C. R. Acad. Sci. Paris 228, 1472–1474 (1949).

This is an expository note on the advantages of the use of the pseudo-tensor $\epsilon_{ijk}$. *L. C. Hutchinson.*

## NUMERICAL AND GRAPHICAL METHODS

***Fisher, Ronald A., and Yates, Frank. Statistical Tables for Biological, Agricultural and Medical Research.** 3d ed. Oliver and Boyd, London, 1948. viii+112 pp. 15s.

[The second edition appeared in 1943; these Rev. 5, 207.] Additions to the third edition consist of Table XI$_1$ . . . which gives the modified probit weighting coefficients necessary for dosage mortality tests involving deaths among controls; the inclusion of the 10 per cent points of $z$ and $e^{2z}$ . . .; the extension of Table XXIII of orthogonal polynomials for values of $n'$ from 53 to 75 . . .; the extension of Table XXVI of natural logarithms to cover the range .100 to 1.00. *From the preface.*

***Table of sines and cosines to fifteen decimal places at hundredths of a degree.** National Bureau of Standards. Appl. Math. Ser., no. 5, viii+95 pp. (1949).

This gives 15-decimal values of $\sin x$ and $\cos x$ for $x=0°(0°.01)45°$; the arguments $45°(0°.01)90°$ are given on the right of the pages. Second central differences are given to aid interpolation; these contain at most 8 figures, so that the table gives 8-decimal values of $\sin x$ and $\cos x$ by linear interpolation. An introduction gives a list of other tables to many figures, and has sections on direct and inverse interpolation. This booklet is most convenient in size and will help to popularise the decimal subdivision of the degree, extremely useful in mathematical work, but neglected during the last two or three centuries largely because instruments and observers are traditionally bound to the sexagesimal subdivision. This table is the first which covers its range since the famous and inaccessible table of Briggs and Gellibrand, Trigonometria Britannica [1633]. *J. C. P. Miller* (London).

**Pooler, L. G. Roots of certain transcendental equations.** Math. Tables and Other Aids to Computation 3, 495–496 (1949).

The first 11 roots of $\tan x+x=0$ and the first 10 of $\tan x+2x=0$ are given to 5 decimals.

**Rothman, M. Table of $\int_0^x I_0(x)dx$ for $x=0(0\cdot1)20(1)25$.** Quart. J. Mech. Appl. Math. 2, 212–217 (1949).

The values of the integral are given to 8 significant figures for $x=0(0.1)20$; the integral multiplied by $e^{-x}$ is given to the same accuracy for $x=15(1)25$. In all cases modified second differences and fourth differences (the latter to 1 or 2 significant figures) are tabulated alongside the main entries. The table was first computed from the power series and then recomputed (up to $x=20$) by means of the formula $\int_0^x I_0(x)dx=2(I_1-I_3+I_5-I_7+\cdots)$. The tables were checked by differencing. *A. Erdélyi* (Pasadena, Calif.).

**Hillman, Abraham, and Sherman, Iva. Complex zeros of $Y_0(z)$, $Y_1(z)$, and $Y_1'(z)$.** Math. Tables and Other Aids to Computation 3, 351–352 (1949).

For the integral order Bessel functions of the second kind, $Y_n(z)$, there are real zeros only on the branch which is real along the positive real-axis. (These zeros are positive.) Every other branch has nonreal zeros in the right half-plane and all branches have nonreal zeros in the left half-plane. In this paper there are listed the first fifteen (nonreal) zeros of $Y_0(z)$, $Y_1(z)=-Y_0'(z)$ and $Y_1'(z)$ in the left half-plane on the branch which has positive real zeros. Approximations to the first few zeros tabulated were obtained from contour lines of $Y_0(z)$ and $Y_1(z)$. These approximations were refined by Newton's iteration method. Zeros with absolute values greater than 10 were computed by the well-known asymptotic expansions for these zeros [Watson, A Treatise on the Theory of Bessel Functions, 2d ed., Cambridge University Press, 1944, in particular, pp. 505–507; these Rev. 6, 64]. *S. C. van Veen* (Delft).

***Tables of the confluent hypergeometric function**

$$F\left(\frac{n}{2},\frac{1}{2};x\right)$$

**and related functions.** National Bureau of Standards. Appl. Math. Ser., no. 3, xxii+73 pp. (1949).

The tables give $F(\frac{1}{2}n,\frac{1}{2},x)$, the solution of

$$xF''+(\tfrac{1}{2}-x)F'-\tfrac{1}{2}nx=0,\quad F(0)=1,\quad F'(0)=n,$$

or the auxiliary functions $F(\frac{1}{2}n,\frac{1}{2},x)/\cosh\{(2n-1)x\}^{\frac{1}{2}}$ or $\{\log F(\frac{1}{2}n,\frac{1}{2},x)\}/\{2nx\}^{\frac{1}{2}}$ for $n=3(2)201$, $x=0(0.01).6(0.05, 0.1 or 0.2) 7(1)45 (5)100$, values being to 6 decimals (6 or 7 significant figures). An introduction by W. Horenstein gives useful properties of the function tabulated, including several asymptotic developments. There are also six pages of interpolation charts indicating the proper Lagrangian formula to be used in various regions of $n$ and $x$. Some small errors, which should not give rise to any difficulty, have been found in the introduction; they are dealt with in more detail in a review of these tables, to appear shortly in Math. Gaz.

As shown in equation (32) of the introduction the function

$$v(y)=e^{-\frac{1}{2}y^2}F(\tfrac{1}{2},\tfrac{1}{2}n,\tfrac{1}{2},y^2)=e^{\frac{1}{2}y^2}F(\tfrac{1}{2}-\tfrac{1}{2}n,\tfrac{1}{2},-y^2)$$

satisfies Weber's equation $v''+(1-2n-y^2)v=0$ from which may be derived relations such as

$$F\left(\frac{2k+1}{2},\frac{1}{2},x^2\right)=\frac{e^{x^2}}{2^k\cdot1\cdot3\cdot\cdots\cdot(2k-1)}H_{2k}^{*}(x)$$

with $H_{2k}^{*}(x)=e^{-x^2}d^{2k}e^{x^2}/dx=(-1)^kH_{2k}(x)$, a Hermite polynomial of imaginary argument. *J. C. P. Miller.*

**Mitchell, K. Tables of the function**

$$\int_0^x \frac{-\log|1-y|}{y}dy,$$

**with an account of some properties of this and related functions.** Philos. Mag. (7) **40**, 351–368 (1949).

Two tables of values of the integral in the title are given; one for the range $x=-1.00(0.01)1.00$ (9D) and the other for the range $x=0.000(0.001)0.500$ (9D). The author obtains various addition formulas which may be used to compute the value of the integral for any real value of $x$ from the values in the second table ($0\leqq x\leqq 0.5$). A discussion of the real and imaginary parts of the integral

$$f(z)=\int_0^z -\zeta^{-1}\log(1-\zeta)d\zeta$$

for complex values of $z$ (with a cut on the real axis of the complex plane from $+1$ to $+\infty$) leads to the evaluation of some trigonometric integrals in terms of the tabulated values of $f(x)$. The author also introduces a generalized function (resembling the Riemann $\zeta$-function),

$$\zeta(a, s/z)=\sum_{n=0}^{\infty} z^{n+1}/(a+n)^s,$$

and shows that certain integrals involving $f(z)=\zeta(1, 2/z)$ may be evaluated in terms of these $\zeta$-functions.

*M. C. Gray* (Murray Hill, N. J.).

**Fletcher, Alan. Guide to tables of elliptic functions.** Math. Tables and Other Aids to Computation **3**, 229–281 (1948).

This comprehensive survey indicates not only all tables but also all known errata, including many previously unpublished, discovered in the author's personal investigations. In the title, the term "elliptic functions" is to be understood in a general sense, as including elliptic integrals and theta functions, because the existing tables relate largely to elliptic integrals. The guide is divided into three parts. Part I is a descriptive guide to all tables considered, referring, with constant obvious abbreviations, to material listed in the detailed bibliography of part II [nearly 180 titles]. Part III is devoted to lists of errata in certain tables marked with an asterisk in part II. Great care has been taken in part III to indicate the nature of the verifications made and the status of the error lists given. *S. C. van Veen.*

**Nelšuler, L. Ya. On optimal threefold triple-entry tables of functions of three variables.** Doklady Akad. Nauk SSSR (N.S.) **64**, 763–766 (1949). (Russian)

The author continues his investigations on the tabulation of functions of three or more variables. Representation in the form $F(x, y, z)=\Phi[f_1(x, y), f_2(x, y)]$ is considered in this paper. Conditions are derived under which this representation is not only possible but optimal in a special sense. No examples are given. [Cf. Uspehi Matem. Nauk (N.S.) **3**, no. 6(28), 205–210 (1948); same Doklady (N.S.) **61**, 791–793 (1948); **60**, 965–968 (1948); Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1169–1191; these Rev. **10**, 149, 359.] *D. H. Lehmer* (Berkeley, Calif.).

**Fréchet, Maurice. Sur une expression simple approchée de la loi de probabilité des erreurs d'observation.** J. Soc. Statist. Paris **84**, 52–69; discussion, 70 (1 plate) (1943).

The author considers various approximations to the normal cumulative distribution function $\Phi(y)=(2\pi)^{-\frac{1}{2}}\int_0^y e^{-\frac{1}{2}u^2}du$, which have the following simple form:

$$1-e^{-ly-my^2}=1-10^{-\gamma y-\delta y^2}.$$

Several criteria of what is a good approximation are considered to decide on the parameters $l$, $m$, $\gamma$ and $\delta$. One method is to let $M_k=M_k'$ for $k=1, 2$, where

$$M_k=2\int_0^\infty y^k d\Phi(y), \quad M_k'=\int_0^\infty y^k d[1-10^{-\gamma y-\delta y^2}].$$

Another is to minimize $V=\int_0^\infty\{\Phi(y)-(1-10^{-\gamma y-\delta y^2})\}^2dy$. There are several other so called empirical methods. The first method above gives $\gamma=.326344$, $\delta=.1731525$, $l=.7514349$ and $m=.3986983$. The second one gives $\gamma=2^{-1}(.4516663)$, $\delta=\frac{1}{2}(.3626359)$, $l=2^{-1}(1.04)$ and $m=\frac{1}{2}(.835)$. Tables are constructed to compare the approximations with the normal distribution. *H. Chernoff* (Urbana, Ill.).

* **Description of a Relay Calculator.** By the Staff of the Computation Laboratory. Harvard University Press, Cambridge, Mass., 1949. xviii+366 pp. $8.00.

After his success with the Automatic Sequence Controlled Calculator, which was the first high-speed digital calculator to provide results in a form accessible to the public, Aiken conceived an improved machine, using relays designed by himself and his staff. This machine, known as Mark II, was built and tested at Harvard, and in 1948 moved to its present location at the Naval Proving Ground, Dahlgren, Virginia. Aiken has accumulated a vast store of knowledge of computing technique by applying the machine built in collaboration with I.B.M. and this logical successor; it may not be too much to say that others have been engrossed with electronic machines before acquiring experience of high-speed computing. Aiken's wisdom is summed up in, "Unless the amount of computation involved in the solution of a problem is relatively great, the saving of time and human labour by automatic calculation may be outweighed by the time and labour required to prepare the problem for the machine and its solution;" and, "Checks, usually of a mathematical nature, must be devised to disclose . . . failures as soon as possible after they occur."

For the technical details of the machine and of its adding and sequence mechanisms and control panels, and auxiliary units for multiplication, square roots, reciprocals and some elementary transcendental functions, the book itself must be consulted. Suffice it to say that there are 13,000 electromechanical relays, some of which are mechanically self-latching and require no power to hold them in position, or if the current fails. Each number contains up to 10 decimal digits and (as in the Bell Telephone Laboratories relay computer) a power between $\pm 15$. The machine is in two similar halves, which may work separately or conjointly. Once every second a complete pattern of 60 impulses of various kinds is repeated, defining the basic cycle of the machine. It is stated that a rough estimate of the time required for the solution of any problem (by one half of the calculator) may be obtained by counting the number of multiplications to be performed, and assuming that each requires half a second. The two proved and tested Harvard relay calculators, which have been well described and are not difficult for those skilled in the art of computing to follow, mark a real beginning of "high-speed computing for all." *L. J. Comrie* (London).

**Wilkes, M. V. Programme design for a high-speed automatic calculating machine.** J. Sci. Instruments **26**, 217–220 (1949).

The author is at present in charge of the development and construction of an automatic computing instrument, the EDSAC (Electronic Delay Storage Automatic Calculator), at the Mathematical Laboratory, University of Cambridge. In this paper he enumerates some of the basic orders for his machine and gives a brief discussion of a programming procedure. This method of programming involves the use of flow-diagrams and seems to be similar to a system already discussed [cf. Goldstine and von Neumann, Planning and Coding of Problems for an Electrical Computing Instrument, The Institute for Advanced Study, Princeton, N. J., 1947; part II, v. II, 1948; part II, v. III, 1948; these Rev. **9**, 208, 622; **10**, 329]. The EDSAC uses standard teleprinter tapes for its input and uses a teleprinter for printing out its final results. The paper closes with a short discussion of a method for modifying orders by arithmetical operations on the so-called address (the reference to a memory position) portion of the order. *H. H. Goldstine* (Princeton, N. J.).

**Booth, Andrew D. A magnetic digital storage system.** Electronic Engrg. **21**, 234–238 (1949).

**Kobrinskiĭ, N. E., and Lyusternik, L. A. Mathematical technics.** Uspehi Matem. Nauk (N.S.) **1**, no. 5–6(15–16), 3–26 (1946). (Russian)

This article forms an introduction to a series of papers [some translations] on computation by physical devices. It is largely concerned with general and rather well-known observations on the philosophy of these devices and such matters as their classification, their relation to and influence on other fields, sources of error, and historical material. Descriptions of and allusions to specific devices supporting the statements made make up much of the discussion of which roughly one-third concerns digital equipment, mostly of the punched-card type, while the remainder is divided among nondigital devices employing mechanical, electrical and optical principles. *R. Church* (Annapolis, Md.).

**Proško, V. M. Apparatus for the determination of the roots of a system of linear equations.** Uspehi Matem. Nauk (N.S.) **1**, no. 5–6(15–16), 41–112 (1946). (Russian)

The author gives a rather complete survey [to the year 1941] of apparatus designed primarily for the indicated purpose. The introductory section is partly concerned with numerical solution; this offers one way of classifying physical methods. It is pointed out that for linear equation solvers the well-known method of improving a solution is very important. The first chapter is devoted to some fifteen mechanical devices of which six are discussed in considerable detail. There follow complete details for four hydraulic methods. The third chapter, on electrical apparatus, occupies about half of the paper and presents the contributions of twelve authors, five being only briefly mentioned. Each chapter has a section giving comparisons and conclusions. Among mechanical devices Wilbur's is preferred while among electrical devices that of Mallock is considered best with the author's own devices [Trav. Inst. Math. Stekloff **20**, 117–128 (1947); these Rev. **10**, 69] receiving favorable mention due to lower cost with comparable accuracy expected. The discussion of each device is independent of the rest of the paper and is usually accompanied by a description of the steps in its operation and comments on accuracy, convenience of application, etc. It appears that the author's dissertation [1940], to which reference is often made, contains much more detail than the present paper. In the bibliography of 70 items, 20 of the 61 which are actually cited are Russian and 7 of these are patents. Thus, although the paper is expository in character, it presents some material which is otherwise less accessible.

*R. Church* (Annapolis, Md.).

**Sadovskiĭ, L. E. Integrating mechanisms.** Uspehi Matem. Nauk (N.S.) **3**, no. 3(25), 113–151 (1948). (Russian)

After starting with the ordinary polar, linear and hatchet planimeters and the Ott integrimeter and without any discussion of moment planimeters the usual classification and examples of harmonic analyzers are given. Product integrators [J. A. Van den Akker] and Stieltjes integrators [E. J. Nystrom] receive attention before turning to integraphs [H. Adler-L. A. Ott] which in turn lead naturally to devices designed to integrate special differential equations [D. M. Myers] and thus to the differential analyzer. The general mathematical theory of this, largely following C. E. Shannon [J. Math. Physics **20**, 337–354 (1941); these Rev. **3**, 279] is outlined. Historical remarks are frequent throughout. Descriptions are brief and encourage the reader to consult the original sources. In view of its pioneering character, a translation of practically all of the paper of G. Coriolis [J. Math. Pures Appl. (1) **1**, 5–9 (1836)] is included as an appendix. *R. Church* (Annapolis, Md.).

**Strobel, Christian. Elektrische Darstellung mathematischer Funktionen.** Arch. Elektrotechnik **34**, 334–338 (1940).

Arbitrary algebraic functions of one independent variable are solved by means of multiple bridge-networks.

*G. Kron* (Schenectady, N. Y.).

**Panov, D. Yu. On a generalization of Bairstow's formula.** Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 331–332 (1949). (Russian)

The author gives a method for an approximate factorization of polynomials of even degree in the form

$$\lambda^{2n}+a_1\lambda^{2n-1}+a_2\lambda^{2n-2}+\cdots+a_{2n}\approx\prod_{k=1}^{n}(\lambda^2+p_k\lambda+q_k).$$

The $p_k$ and $q_k$ are rational functions of the $a_j$ but the explicit expression is not given. When $n=2$, the author's method gives Bairstow's formula

$$\lambda^4+B\lambda^3+C\lambda^2+D\lambda+E\approx(\lambda^2+B\lambda+C)\left(\lambda^2+\frac{CD-EB}{C^2}\lambda+\frac{E}{C}\right).$$

*A. W. Goodman* (Lexington, Ky.).

**Pflanz, Erwin. Zur Bestimmung reeller Nullstellen von reellen Funktionen einer Variablen.** Z. Angew. Math. Mech. **29**, 85–91 (1949). (German. Russian summary)

The author investigates further the case $m=2$ of his former paper [same Z. **28**, 114–122 (1948); these Rev. **9**, 621] when $f''(x)$ is monotonic and $f(x_1)f(x_3)<0$. While in general the zeros of the two parabolas which replace and include $f(x)$ converge cubically towards the root $X$, a quadratic function $q(x)$ can be determined such that for $v(x)=q(x)f(x)$ the convergence is of the fifth degree. In another supplement the case when $f''(x)$ is not monotonic is reduced to the former case. *E. Bodewig* (The Hague).

**Rubbert, P. K. Beitrag zur inversen Interpolation.** Z. Angew. Math. Mech. **29**, 93–94 (1949).

If $x'''(y)$ does not change sign in the interval $(y_{-1}, y_2)$, then the following quadratic formulae hold:

$$x(y) \approx x_0 + Y_0\left\{1 - a\frac{\Delta y_0 - \Delta y_1}{\Delta y_0 \Delta y_1}\right\},$$

$$x(y) \approx x_0 + Y_0\left\{1 - a\frac{\Delta y_{-1} - \Delta y_0}{\Delta y_{-1}\Delta y_0}\right\},$$

where $Y_0 = \omega(y-y_0)/\Delta y_0$, $a = (y_1-y)/2$ and the true $x$ lies between these two values. The error of the linear formula $x(y) = x_0 + \omega/(1+A)$, where $A = (y_1-y)/(y-y_0)(\Delta y_0)$, is equal (apart from terms of third degree) and opposite to that of the regula falsi: $x = x_0 + \omega(y-y_0)/\Delta y_0$. A further formula is

$$x = x_0 + \frac{y-y_0}{y_1-y_0}(\omega + a(y_1'-y_0')/y_0'y_1').$$

[See the same J. **28**, 122–124 (1948); these Rev. **9**, 620.]

*E. Bodewig* (The Hague).

**Zurmühl, Rudolf. Zur numerischen Auflösung linearer Gleichungssysteme nach dem Matrizenverfahren von Banachiewicz.** Z. Angew. Math. Mech. **29**, 76–84 (1949). (German. Russian summary)

The author describes the method of Banachiewicz [Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Sér. A. Sci. Math. **1938**, 393–404 (1939)] for solving linear algebraic equations; it consists essentially in decomposing the matrix $A$ into the product of two triangular matrices with opposite directions: $A = LR$, where $R$ has elements 1 in the diagonal. [The reviewer wishes to emphasize that Banachiewicz's method results by applying the Gaussian method in a special way, namely by computing for every derived system only the first equation and the first column and writing them down, while the remaining coefficients of every other equation of the same derived system are not computed until they come into the first row or column of a derived system. This yields precisely the method, the scheme and the operations of Banachiewicz and of the author, but without seeking the decomposition $A = LR$, although it is yielded automatically by the process. This variant of Gauss's method has a disadvantage which is concealed by the form Banachiewicz and the author give it by starting from $A = LR$, but which appears immediately if it is treated in the above manner. In fact, by choosing by this rigid scheme only the first equation of every derived system, it is possible that the leading coefficient of a derived system is small or even vanishes so that division by it is troublesome or even impossible and the procedure must be modified.]

*E. Bodewig* (The Hague).

**Lohman, John B. An iterative method for finding the smallest eigenvalue of a matrix.** Quart. Appl. Math. **7**, 234 (1949).

If $\lambda$ is an approximation to the smallest eigenvalue of $A$, the author determines the largest eigenvalue of $B^{-1}$, where $B = A - \lambda E$. If also the other eigenvalues of $A$ are known approximately, the method can be improved by determining the largest eigenvalue of $B^{-1} - aE$, where $a$ is suitably chosen. [The author's method requires many operations; their number can be diminished by solving the system of linear equations $BZ_i = Y_i$, where $Y_{i+1} = Z_i/z_n$. Then the eigenvalue is approximately $\lambda_i = \lambda + 1/z_n$.] *E. Bodewig.*

**Stange, K. Mehrfaches Ausgleichen einer fehlerhaften Punktreihe.** Z. Angew. Math. Mech. **29**, 114–126 (1949). (German. Russian summary)

The process of the repeated graduation of a set of points $x_k(l_k)$, which possess errors, by means of a power series $\sum_{\nu=-s}^{s} \lambda_\nu x_{k+\nu}$, where the $\lambda_\nu$ are the weights, is ineffective after a few steps. This is due to the fact that the correlation between the errors of the smoothed data increases with each step. The calculation for linear and quadratic graduation is given and recurrence formulas for the weights, the moments of the weights, the error, and the correlation function. The author also shows the influence of a correlation of initial errors on the propagation of the errors. *E. Frank.*

**Kreis, H. Über eine Erweiterung des Ausgleichungsverfahrens von Karup.** Mitt. Verein. Schweiz. Versich.-Math. **49**, 120–130 (1949).

By integrating certain interpolation curves which satisfy preassigned conditions concerning their derivatives the author obtains a set of graduation formulae which can be regarded as a generalisation of Karup's graduation formula.

*P. Johansen* (Copenhagen).

**Tchakaloff, L. Über eine allgemeine Quadraturformel.** C. R. Acad. Bulgare Sci. Math. Nat. **1**, no. 2–3, 9–12 (1948).

Der Verf. stellt sich zur Aufgabe eine allgemeine Quadraturformel der Gestalt

$$(1)\qquad \int_a^b f(x)dx = \sum_{k=1}^{m}\sum_{\lambda=0}^{\nu_k} A_{k\lambda} f^{(\lambda)}(a_k), \quad a \leqq a_1 < \cdots < a_m \leqq b, \quad \nu_1, \cdots, \nu_m \geqq 0,$$

zu bestimmen. Die von $f(x)$ unabhängigen Koeffizienten $A_{k\lambda}$ (deren Anzahl $\sum_{k=1}^{m}(\nu_k+1) = M+1$) sind so zu bestimmen, dass (1) für jedes Polynom $f(x)$ $M$-ten oder niederen Grades bestehen bleibt. Es stellt sich heraus, dass diese Aufgabe eine eindeutig bestimmte Lösung besitzt. Die angewandte Methode gestattet die Werte der Koeffizienten $A_{k\lambda}$ effektiv zu bestimmen in der folgenden einfachen Weise: Mittels des Polynoms $(M+1)$-ten Grades

$$P(x) = \prod_{k=1}^{m}(x-a_k)^{\nu_k+1}$$

wird die rationale Funktion

$$R(z) = \frac{1}{P(z)}\int_a^b \frac{P(z)-P(x)}{z-x}dx$$

gebildet. Die Koeffizienten $A_{k\lambda}$ werden bestimmt durch die Partialbruchzerlegung $R(z) = \sum_{k=1}^{m}\sum_{\lambda=0}^{\nu_k} B_{k\lambda}(z-a_k)^{-\lambda-1}$; $A_{k\lambda} = B_{k\lambda}/\lambda!$. Für die vollständige Entwicklung mit Restglied von $f(x)$, die im Intervall $a \leqq x \leqq b$ definiert und dort mit stetigen Ableitungen bis zur $(M+1)$-ten Ordnung versehen ist, findet der Verf.:

$$\int_a^b f(x)dx = \sum_{k=1}^{m}\sum_{\lambda=0}^{\nu_k} A_{k\lambda} f^{(\lambda)}(a_k) + \int_a^b \left\{\frac{(b-t)^{M+1}}{(M+1)!} - u(t)\right\} f^{(M+1)}(t)dt$$

wobei $u(t) = \sum_{k=1}^{m}\sum_{\lambda=0}^{\nu_k} A_{k\lambda}(a_k-t)^{M-\lambda}/(M-\lambda)!$ für $a_{s-1} \leqq t \leqq a_s$ $(s = 1, 2, \cdots, m)$, $a_0 = a$; $u(t) \equiv 0$ für $a \leqq t < a_{s-1}$, $a_s < t \leqq b$.

*S. C. van Veen* (Delft).

**Wilson, E. M. A note on the numerical integration of differential equations.** Quart. J. Mech. Appl. Math. **2**, 208–211 (1949).

The method used here is an adaptation of the Taylor series method for step-by-step integration:

$$(1)\quad y^{(k)}(x+w)=y^{(k)}(x)+wy^{(k+1)}(x)+(w^2/2!)y^{(k+2)}(x)+\cdots.$$

For a differential equation of order $m$, one uses the $m$ series (1) corresponding to $k=0, 1, \cdots, (m-1)$. Theoretically, these series must be evaluated at each step of the integration. The author's version shifts this burden of computation to a different stage of the program, concentrating it in the preliminary calculation of certain auxiliary functions. These then become the coefficients in linear combinations which give the value of the unknown function and its first $m-1$ derivatives at a point of tabulation in terms of their values at the preceding tabular point. For example, taking $k=0$ in (1), the solution of the first-order linear equation $y'=P_1(x)y+Q_1(x)$ is obtained from

$$y(x+w)=P(x, w)y(x)+Q(x, w),$$

where $P(x, w)=1+wP_1(x)+(w^2/2!)P_2(x)+\cdots$ and

$$Q(x, w)=wQ_1(x)+(w^2/2!)Q_2(x)+\cdots.$$

The $P_j$ and $Q_j$ are given, for $j\geqq 2$, by the recurrence formulas $P_{r+1}=P_r'+P_1P_r$, $Q_{r+1}=Q_r'+Q_1P_r$ which one obtains by repeated differentiation of (and substitution from) the original differential equation upon writing

$$d^ry/dx^r=P_r(x)y+Q_r(x).$$

The tabular step $w$ having been decided upon, one now calculates $P(x, w)$ and $Q(x, w)$ (preferably using automatic calculating machines such as punched-card, or National accounting machines) for the tabular values of $x$ desired. Then the actual step-by-step part of the computation involves only a single multiplication and addition for each tabular entry. Presumably one may, for example, use the present method to advantage over the customary Taylor series method in the case of solving the same differential equation for several different sets of initial conditions. The author illustrates the method as applied to a linear second-order equation. *G. B. Thomas, Jr.* (Cambridge, Mass.).

**Dainelli, Dino. Sull'integrazione numerica delle equazioni differenziali ordinarie.** Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 393–405 (1948).

This article contains a variation of the well-known Adams-Störmer step-method for the numerical integration of ordinary differential equations with given initial conditions. The scheme of computation is simplified in some respects. The author establishes the convergence of the process if the number $n$ of steps in a given finite interval of integration approaches infinity. The method is illustrated by the integration of $y'''=\frac{1}{2}(q+2x^3)-4y^2-2y'^2-2xy''$ (initial conditions $y(0)=1$; $y'(0)=0.5$; $y''(0)=-2$) in the interval (0, 1), with $n=10$. The results appear to be in agreement with the exact values to 4 or 5 figures. *S. C. van Veen* (Delft).

**Fox, L., and Goodwin, E. T. Some new methods for the numerical integration of ordinary differential equations.** Proc. Cambridge Philos. Soc. **45**, 373–388 (1949).

The authors are concerned with numerical methods of solving the initial value problem for ordinary differential equations (linear or nonlinear, mainly of first and second order) with the types (i) $y'=fy+g$; (ii) $y''+fy'+gy=0$ discussed in particular. The method is a finite difference step-by-step integration at equidistant $x_i$ ($h$ apart) in which the equation (say (i)) is replaced by a simple recurrence linking 2 or 3 consecutive values $y_i=y(x_i)$ and a remainder term $\Delta$ involving high order differences of the $y(x)$ sought. E.g., for (i) the authors (method III) write

$$(iii)\quad (1-\tfrac{1}{3}hf_1)y_1=\tfrac{4}{3}hf_0y_0+(1+\tfrac{1}{3}hf_{-1})y_{-1}+\tfrac{1}{3}h(g_1+4g_0+g_{-1})+\Delta,$$

where

$$(iv)\quad \Delta=(-\tfrac{1}{90}\mu\delta^5+\tfrac{1}{8}\tfrac{1}{75}\mu\delta^3-\cdots)y_0.$$

The recurrence (iii) is used to compute $y_1$ from $y_0$ and $y_{-1}$ and known quantities (ignoring $\Delta$), and the process carried over the range of integration to obtain the trial $y_i$. These are then substituted in $\Delta$ for the second correction run, the process being iteratively repeated if necessary. While with the classical methods the $y_i$ would be adjusted for $\Delta$ after one or two steps, this adjustment is here postponed to a systematic second run, and for this purpose the remainder $\Delta$ should be small and yet the recurrence formula simple. Seven particularly convenient formulae (such as (iii), (iv)) are provided and their relative merits illustrated with examples. The question arises whether it is profitable to iterate after the trial $y_i$ is carried over a fraction of the range only, but this is not discussed. *H. O. Hartley.*

**Tasny-Tschiassny, L. The triangulation of a two-dimensional continuum for the purpose of the approximate solution of second-order partial differential equations.** J. Appl. Phys. **20**, 419–424 (1949).

The author considers the boundary value problem of a second order partial differential equation of the type (1) $(\gamma_1v_x+\gamma_3v_y)_x+(\gamma_3v_x+\gamma_2v_y)_y-i(x, y)=0$ and sets out to generalize the finite difference-relaxation method for any irregular triangular grid. A finite process of solution is described in which certain quantities are described only by physical analogy and which is said to converge to the solution of (1) with refinement of the grid, but no remainder or correction term is given. An example considered by Gandy and Southwell is reworked, and it is shown (although full details are not given) that with grid refinement the finite solution is within 2% of the exact one.

*H. O. Hartley* (London).

**Zeilon, Nils. Sul calcolo numerico degli autovalori.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 52–60 (1949).

Let the kernel $K$ of the integral equation $\varphi=\lambda K\varphi$ be symmetric and positive definite. Let $\varphi_1, \varphi_2, \cdots$ be the orthogonal eigenfunctions, and $v_0$ an approximation of $\varphi_1$ so that $c_1$ in $v_0=c_1\varphi_1+c_2\varphi_2+\cdots$ predominates. With $v_1=Kv_0$, $v_2=Kv_1$, $\cdots$ one has the well-known sequence $K_0=1$, $K_1=\sum c_i^2/\lambda_i$, $K_2=\sum c_i^2/\lambda_i^2$, $\cdots$ and the sequence $\lambda^{(i)}=K_i/K_{i+1}$ which decreases monotonically to $\lambda_1$. The author's aim is to determine a second sequence $\mu^{(i)}$ which increases monotonically to $\lambda_1$. He forms the sequence $q^{(i)}=\Delta\lambda^{(i)}/\Delta\lambda^{(i+1)}$ and proves first that the sequence

$$\lambda(q^{(i)})=[q^{(i)}\lambda^{(i+1)}-\lambda^{(i)}]/(q^{(i)}-1)$$

converges more rapidly to $\lambda_1$, though it is still monotonically decreasing. Second, for arbitrary fixed $s$ the sequence $\mu^{(i)}=[\lambda(q^{(i)})-s\lambda^{(i+1)}]/(1-s)$ has at most a single minimum, so that it increases monotonically to $\lambda_1$ thereafter. Moreover, the $q^{(i)}$ converge to $\lambda_2/\lambda_1$. *E. Bodewig.*

**Zwinggi, Ernst. Variation der Rechnungsgrundlagen in der Invalidenversicherung.** Mitt. Verein. Schweiz. Versich.-Math. **49**, 158–164 (1949).

It is always laborious to calculate premiums for disability insurance. The author studies a method of approximating the premiums in the case when no recoveries are assumed, the mixed life table is substituted for the active life table, and the mortality of disabled lives is assumed to be the same as the mortality of mixed lives. The probability of becoming disabled is assumed to be given by Makeham's formula $i_x=\alpha+\beta c^x$, where $x$ denotes the age. The difficulties are connected with the exponential expression $c^x$ and the approximate solution is obtained by putting $c=c_0+\lambda$, where $c_0$ is a constant and $\lambda$ a new parameter. If $c_0=1.1$ then $|\lambda|$ will in practice never exceed 0.02. *K. Medin* (Uppsala).

**Ruch, H. Zum Zinsfussproblem.** Mitt. Verein. Schweiz. Versich.-Math. **49**, 131–138 (1949).

It is well known that life insurance premiums decrease when the rate of interest increases, in the case where the mortality does not decrease with increasing age. The author shows that the same is true to a certain extent independently of the behavior of the force of mortality.

*P. Johansen* (Copenhagen).

## ASTRONOMY

**Stumpff, K. Neue Formeln und Hilfstafeln zur Ephemeridenrechnung.** Astr. Nachr. **275**, 108–128 (1947).

Expressions for the coordinates in the two-body problem (frequently used in ephemeris calculation and methods of orbit determination) are

$$x=x_0F(\tau)+x_0'G(\tau),$$
$$y=y_0F(\tau)+y_0'G(\tau),$$
$$z=z_0F(\tau)+z_0'G(\tau),$$

where $x, y, z$ are the rectangular coordinates and $x', y', z'$ the velocity components at time $t$ and $x_0, y_0, z_0, x_0', y_0', z_0'$ the corresponding "local elements" at time $t_0$, with $\tau=k(t-t_0)$. The functions $F$ and $G$ depend on $\tau$ and, in addition, on these local elements. The latter occur in combinations that are invariant under coordinate transformations. As three independent local invariants the author chooses

$$\mu=r_0^{-3},\quad \sigma=r_0'r_0^{-1},\quad \omega=(x_0'^2+y_0'^2+z_0'^2)r_0^{-3},$$

where $r_0$ is the radius vector at $t_0$. Combinations of these, such as

$$\epsilon=\omega-\mu,\quad \rho=\mu-\epsilon,\quad \vartheta=\omega-\sigma^2$$

are introduced. All functions of the coordinates that are independent of the orientation of the coordinate system can be expressed in terms of these quantities.

Let $E_0$ be the eccentric anomaly at $t_0$, $E$ that at $t$, and

$$E=E_0+\lambda,\quad \eta=\sigma\tau,\quad \zeta=\epsilon\tau^2,\quad \chi=\rho\tau^2,\quad z=\lambda\chi^{-\frac{1}{2}}.$$

It is shown that $z$, not to be confused with the coordinate of the same designation, satisfies the transcendental equation

$$z+c_3(\chi z^2)\cdot\eta z^2+c_4(\chi z^2)\cdot\zeta z^2=1.$$

The functions $c_\nu(\lambda^2)$ differ by a factor $\lambda^{-\nu}$ from the functions $C_\nu(\lambda)$ used in a later article [see the following review]. This equation replaces Kepler's equation and is introduced as the principal equation for ephemeris calculation. After a numerical solution for $z$ has been found, the functions $F$ and $G$ as well as their derivatives are obtained from simple expressions.

The later pages of the article are occupied with various applications, a set of tables of the function $c_0, c_1, c_2, c_3$, and tables for the solution of the principal equation. The method appears to be eminently suitable for ephemeris calculations.

*D. Brouwer* (New Haven, Conn.).

**Stumpff, K. Über die Reihenentwicklung der rechtwinkligen Bahnkoordinaten im Zweikörperproblem. II.** Astr. Nachr. **275**, 203–222 (1947).

This is closely related to a previous article by the same author [Astr. Nachr. **274**, 49–68 (1943); these Rev. **7**, 224]. An important part of this communication is a study of the functions $C_\nu(\lambda)$ and $c_\nu(\lambda)$, defined by $C_0(\lambda)=\cos\lambda$, $C_\nu(\lambda)=\int_0^\lambda C_{\nu-1}(\lambda)d\lambda$, $c_\nu(\lambda)=(\nu!/\lambda^\nu)C_\nu(\lambda)$. With the aid of these functions expansions are given of the functions $F(\tau)$ and $G(\tau)$ which arise in connection with the expressions for the rectangular coordinates in the two-body problem for any date. The literal developments so obtained are shown to have practical advantages only for orbits with very small eccentricities. *D. Brouwer* (New Haven, Conn.).

**Stumpff, K. Über die Eulersche Gleichung und verwandte Beziehungen in der Theorie der Parabelbewegung.** Astr. Nachr. **276**, 164–168 (1948).

In an earlier paper [see the second preceding review] the author showed that all geometrical relations in unperturbed motion that do not depend on the orientation of the coordinate system may be expressed with the aid of three independent "local invariants." In this paper a derivation of Euler's equation in parabolic motion is given with the aid of these local invariants. This is followed by a discussion of a similar equation involving the velocities at two different points in a parabolic orbit. This leads to some apparently new geometrical relations among two points in a parabolic orbit and the velocities at these points. *D. Brouwer.*

**Jardetzky, W. Einige Bemerkungen zur Einführung neuer Elemente in der Bahnbestimmung.** Astr. Nachr. **275**, 223–228 (1947).

The vectorial orbital elements, introduced by M. Milankovitch, and the "local elements" used by K. Stumpff in a series of papers [cf. the three preceding reviews] are shown to be related, and some of the results obtained by Stumpff are derived with the aid of Milankovitch's vector relations. By using a method of reversion of power series the solution of Stumpff's principal equation for ephemeris calculation is obtained. *D. Brouwer* (New Haven, Conn.).

**Bucerius, H. Zur Fourier-Analyse der Lösungen des Zweikörperproblems.** Astr. Nachr. **275**, 193–202 (1947).

The reciprocal of the radius vector in hyperbolic motion is expanded with the aid of the Fourier integral theorem. The functions that arise in this expansion are Hankel functions with purely imaginary index and argument. These functions appear to have a significance in hyperbolic motion analogous to that of the Bessel functions with real index and argument in elliptic motion. Asymptotic methods are used to establish the convergence. The cases of near-parabolic, parabolic, and rectilinear motion are also treated.

*D. Brouwer* (New Haven, Conn.).

***Okyay Kabakcioğlu, T. Berechnung der langperiodischen Störungen 2. Ordnung der Massen bei mehrfach kommensurablen Typen.** Université d'Istanbul. Faculté des Sciences. Recueil de mémoires commémorant la pose de la première pierre des Nouveaux Instituts de la Faculté des Sciences, pp. 72–77, Istanbul, 1948.

This investigation is a further development of an earlier contribution by the author [Rev. Fac. Sci. Univ. Istanbul. Ser. A. 6, 192–223 (1941); these Rev. 4, 259] containing a generalization of a method due to A. Wilkens for dealing with critical long-period inequalities in planetary theory. The present developments deal in detail with the evaluation of perturbations in the elliptic elements of the second order as to the perturbing masses for a selected argument.

*D. Brouwer* (New Haven, Conn.).

**Bilimovitch, Anton. Ueber einen Sonderfall des Vierkörperproblems.** Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. 7, 213–227 (1941).

The four masses are taken pairwise equal and each pair is initially symmetrically located and possesses symmetric initial velocities with respect to a line through the center of mass of the system. The law of force $k^2mm'r^p$ is assumed and several integrals are obtained. Some special cases are considered and in particular the problem is solved completely for the case of all four masses equal and equidistant from the line of symmetry, the lines of centers of each pair mutually perpendicular, the initial velocities symmetric with respect to the center of mass but with no angular velocity, and $p=-3$. There are several misprints: in equation (1) the subscript on the final $\rho$ should be 4; in the two equations at the top of page 225 the negative exponents should be $-4$ throughout. *R. G. Langebartel.*

**Bilimovitch, Anton. Ueber die Rolle der Archimedischen gleicheckigen Polyeder im $n$-Körperprobleme.** Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. 7, 197–212 (15 plates) (1941).

The article contains a brief catalog of the fifteen Archimedean polyhedra [cf. M. Brückner, Vielecke und Vielflache, Teubner, Leipzig, 1900, pp. 132–140] which are suggested as natural configurations to be considered in the $n$-body problem in view of symmetries present and the equidistance of the vertices from the center. The case for one of these polyhedra, namely the polyhedron with eighteen equal edges and twelve vertices arising from cutting off the corners of a regular tetrahedron, is carried out in some detail. With the initial velocities directed toward (or directly away from) the center and the law of force $km^2r^p$ assumed, the author shows that the configuration of twelve bodies of equal mass is maintained only for $p=1$ or $p=3$. The formula for $F_s$ on page 208 contains several typographical errors and should read

$$F_s=km^2u^p[\tfrac{3}{4}\sqrt{3}+2^p\sqrt{3}+(\sqrt{3})^{p+2}+\tfrac{7}{8}\sqrt{3}(\sqrt{5})^{p-1}].$$

*R. G. Langebartel* (Urbana, Ill.).

**Holopov, P. N. Determination of the spatial density of stars in a spheroidal stellar cluster.** Akad. Nauk SSSR. Astr. Zhurnal 26, 110–114 (1949). (Russian)

A generalization of von Zeipel's solution, following ten Bruggencate's. The surfaces of equal density are supposed to be concentric and similar ellipsoids of revolution. While with ten Bruggencate's method only stars along the minor axis of the ellipsoids could be used, the author develops formulae which permit the use of all the stars in the cluster.

*Z. Kopal* (Cambridge, Mass.).

**Cowling, T. G., and Newing, R. A. The oscillations of a rotating star.** Astrophys. J. 109, 149–158 (1949).

Rayleigh's principle is derived in a form suitable for the study of oscillations of a rotating star about its equilibrium state. This modified form of Rayleigh's principle is derived for a finite number of degrees of freedom, but is easily generalized to apply to any dynamical system. Periods of oscillation of rotating stars may be found from the derived formula when those of the corresponding non-rotating stars are known. With similar approximations the formula gives the earlier formula of Ledoux. By more exact numerical calculations the periods are obtained for certain special cases. Nonradial oscillations are also studied.

*G. Randers* (Oslo).

**Ghosh, N. L. A note on Hamy's theorem.** Bull. Calcutta Math. Soc. 40, 229–230 (1948).

The author shows that the theorem discovered by Hamy [Ann. Observ. Paris 19 (1889)] about the impossibility of an ellipsoidal density-stratification in a heterogenous rotating liquid is more general than originally indicated. The theorem is shown not to depend upon the particular ellipsoidal potential resulting from the rotating liquid itself, but to be valid for any ellipsoidal potential. *G. Randers.*

**Sobolev, V. V. On the distribution of brightness on a stellar disk.** Akad. Nauk SSSR. Astr. Zhurnal 26, 22–27 (1949). (Russian)

The author considers the center-to-limb variation of brightness over the apparent disk of a star which is imbedded in a scattering atmosphere of negligible extent. The plane-parallel equations of radiative transfer are set up and their numerical solutions investigated for the case of scattering which is isotropic, or characterized by a linear scattering indicatrix. The results are briefly compared with previous solutions of the same problem by Ambarzumian [C. R. (Doklady) Acad. Sci. URSS (N.S.) 43, 102–106 (1944)] and Chandrasekhar [Astrophys. J. 100, 76–86, 117–127 (1944); these Rev. 6, 76, 190]. *Z. Kopal.*

## RELATIVITY

**Willmore, T. J. Regraduation in spherically symmetric space-times of general relativity.** Philos. Mag. (7) 40, 428–434 (1949).

La possibilité d'étendre à la relativité générale le processus de regraduation utilisé en relativité cinématique a été examiné par A. G. Walker [Proc. Roy. Soc. Edinburgh. Sect. A. 62, 164–174 (1946); ces Rev. 7, 531]. Au sens de Walker la regraduation ne concerne que les coordonnées locales et laisse inaltérées les coordonnées de base attachées à l'événement envisagé. L'auteur étudie ici les regraduations non uniformes des $ds^2$ à symétrie sphérique. On peut ainsi caractériser les $ds^2$ de Lemaître par la condition que ce

sont les seuls $ds^2$ à symétrie sphérique et pression isotrope qui admettent une regraduation fonction seulement du temps. Inversement toutes les regraduations possibles des $ds^2$ de Lemaître sont recherchés et le cas particulier de l'univers statique d'Einstein étudié. *A. Lichnerowicz.*

**Rabe, E. Zur Singularität der Schwarzschildschen Lösung für $r=2m$.** Astr. Nachr. **275**, 251–255 (1947).

The singularity of Schwarzschild's solution for the gravitational field of a mass point is studied with the intention of obtaining a physical interpretation of the singularity. It is shown that the singularity has a relation to the point in the gravitational field at which a freely falling point particle will attain the velocity of light when falling from the outer limits of the gravitational field. This property of a falling particle is of course not dependent on the coordinates used and the interpretation is therefore valid for any form of Schwarzschild's solution. *G. Randers* (Oslo).

**Finzi, Bruno. Discontinuità sul fronte d'onda delle azioni gravitazionali.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 18–25 (1949).

The characteristics of the fundamental gravitational equations of Einstein, which give the wave-fronts of gravitational propagation, were determined by Levi-Civita [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **11**, 3–11, 113–121 (1930)]. A mistake in Levi-Civita's paper is corrected, and a much simpler argument for the determination of the characteristics is given. *A. J. McConnell.*

**Jordan, P. Fünfdimensionale Kosmologie.** Astr. Nachr. **276**, 193–208 (1948).

Although it is not always easy to follow the author's arguments, this memoir provides a useful summary of investigations by the author and his collaborators on five-dimensional projective relativity and its applications to theoretical cosmology and stellar evolution. The first part, which is self-contained, is devoted to a simplified account of the extension of Veblen's projective relativity which was first developed by the author in 1944. The group of transformations under which the field equations are invariant is assumed to be isomorphic with the group of all homogeneous transformations in five variables $X^\mu$, $\mu=1, 2, \cdots, 5$. A metric $g_{\mu\nu}$ is introduced defining a fundamental invariant $J=g_{\mu\nu}X^\mu X^\nu=X_\nu X^\nu$.

In the second part, which is devoted to the physical interpretation of the mathematical relationships already established, the invariant $J$ is interpreted as a scalar function of the field, and the possibility of its "variation" is shown to provide a precise basis for Dirac's idea of the variation of the gravitational constant with time. [This idea was, in fact, originally suggested by Milne.] The "inductive cosmology" developed in this part is based on what the author calls "Dirac's principle," viz., that the fundamental magnitudes in cosmology can be expressed as multiples of the following microphysical units: $h$, length unit $10^{-13}$ cm., and time unit $10^{-23}$ sec., the associated mass unit being comparable with the proton mass $m_p$, and velocity unit with $c$ [Dirac, Nature **139**, 323 (1937); Proc. Roy. Soc. London. Ser. A. **165**, 199–208 (1938)]. With this particular choice of units, the mass and radius of the universe are expressed as powers of its age, $t$ ($\sim 10^{40}$). Applying Hubble's law, the radius is given by $t$, and the mass ($\sim 10^{80}$) by $t^2$. The "constant" of gravitation is then found to vary as $t^{-1}$. The upper limit of stellar mass ($\sim 10^{60}$) appears to increase as $t^1$, and hence the total number of stars as $t^1$. The total number of spiral nebulae then varies as $t^1$, the average number of stars in each also varying as $t^1$.

The cosmological "constant" in the Einstein universe is equal to the inverse square of the radius, and so varies as $t^{-2}$. Just as this constant must be introduced into the field equations of general relativity to obtain a closed universe, so to obtain a closed model the field equations of projective relativity must also be modified. For full details the author refers to a paper by Ludwig and Müller [Ann. Physik (6) **2**, 76–84 (1948); these Rev. **10**, 408]. He discusses two particular models. In the first, matter is assumed to be static in the large, and the author shows that Ludwig and Müller's results are equivalent to those of his own inductive cosmology. In the second model, matter is assumed to be at so high a temperature that it moves practically with the velocity of light; in this model, the "constant" of gravitation decreases as $t^{-2}$, density is constant, and total mass increases as $t^2$.

Turning to the problem of stellar evolution, the author maintains that the most important task of any theory is to explain the upper limit of stellar mass ($\sim 10^{60}\ m_p$). He criticises the view that this is determined by some criterion of stability. Instead, he suggests that stars originate as separate "sub-universes," of the type of the second model, which "amalgamate" with the main universe when the gravitational "constant" is the same for each, i.e., when $t_1^{-2}=t^{-1}$, where $t_1$ applies to the sub-universe. For the present value of $t$ ($\sim 10^{40}$), this gives $t_1 \sim 10^{20}$, i.e., $10^{-3}$ sec., and hence the mass of the typical star, which is given by $t_1^3$, is of the order of $10^{60}\ m_p$. The author concludes with a short discussion of super-novae and Baade's stellar populations I and II. *G. J. Whitrow* (London).

**Tonini, Valerio. Relatività strutturale.** Rend. Sem. Fac. Sci. Univ. Cagliari **17** (1947), 269–292 (1948).

## MECHANICS

**Kasner, Edward, and De Cicco, John. Geometric theorems in dynamics.** Math. Mag. **22**, 223–233 (1949).

This paper deals with a problem which has been treated by Kasner in previous papers [e.g., Trans. Amer. Math. Soc. **7**, 401–424 (1906)] and in his monograph Differential-Geometric Aspects of Dynamics [Amer. Math. Soc. Colloquium Publ., v. 3, part 2, New York, 1913]. Specifically, the problem is that of determining a set of properties which is necessary and sufficient for a three-parameter family of plane curves to be the family of trajectories of a particle moving in a positional field of force. This paper gives new proofs of some of the previously known theorems, and elaborates further on various aspects of the problem.
*L. A. MacColl* (New York, N. Y.).

**Kasner, Edward, and De Cicco, John. Physical families in conservative fields of force.** Proc. Nat. Acad. Sci. U. S. A. **35**, 419–422 (1949).

In this note the authors continue their discussion of what they call physical families of curves. [For a definition see the same Proc. **34**, 68–72 (1948); these Rev. **9**, 375.] They now consider the case of motion on a plane or other surface

in a conservative field of force, stating various simple basic properties of the families of curves. *L. A. MacColl.*

**Dizioğlu, Bekir. Spezielle $R_M$-Kurven und ihre Anwendungen.** Rev. Fac. Sci. Univ. Istanbul (A) **14**, 49–64 (1949). (German. Turkish summary)

Let a movable plane body assume three different positions in the fixed plane. Let the radius of the circle through corresponding points be $R$. The $R_M$-curves, which were extensively investigated by H. Alt [Z. Angew. Math. Mech. **1**, 373–398 (1921)] and A. E. Mayer [Math. Z. **43**, 389–445 (1937)], are the loci of points which have the same $R$. These curves are tricircular sextics and are special cases of the "three-bar curves" (Koppelkurven) traced by a point on the connecting rod joining two cranks. The triangle of the double points coincides with the triangle of the focal points. Special cases are investigated here including those in which this triangle is equilateral and the three-bar curve has three cusps. Constructions are given for points on the $R_M$-curve. These are applied to the determination of the linkage which makes four given positions of a link correspond to four given positions of a crank. *M. Goldberg* (Washington, D. C.).

**Locatelli, P. Spunti di meccanica analitica nella dinamica delle costruzioni.** Rend. Sem. Mat. Fis. Milano **18** (1947), 124–139 (1948).

**Trăilescu, S. Sur l'accélération du second ordre.** Bull. Sci. Tech. Polytech. Timişoara **13**, 239–245 (1948).

The author derives some relations for relative motion involving the third time derivative. *P. Franklin.*

***Schuler, Max. Mechanische Schwingungslehre. Teil I. Einfache Schwinger.** Wolfenbütteler Verlagsanstalt, Wolfenbüttel und Hannover, 1949. 168 pp. 6 D.M.

This is a brief elementary text on vibrations. Most of the book is concerned with physical problems whose equation is that of the damped pendulum, linearized, free or with a forcing term. A few cases with terms quadratic in the velocity are treated. *P. Franklin* (Cambridge, Mass.).

**Bautin, N. N. On L. I. Mandel'štam's problem in the theory of clocks.** Doklady Akad. Nauk SSSR (N.S.) **65**, 279–282 (1949). (Russian)

Let $\tau$ be the period of a pendulum and $\lambda_1, \lambda_2, \cdots$ various physical parameters upon which it may depend. Mandel'štam proposed the problem of comparing the partials $\partial\tau/\partial\lambda_i$ for clocks of Galileo-Huyghens type and for pre-Galilean clocks (i.e., without pendulum or spring). He suggested that this would bring out clearly the role of pendulum or spring as period stabilizers. This problem is solved by the author for the ideal clock model of his previous note [Doklady Akad. Nauk SSSR (N.S.) **61**, 17–20 (1948); these Rev. **10**, 409]. *S. Lefschetz* (Princeton, N. J.).

**Tkačev, L. I. On the 84-minute period for a system with coupled or free gyroscopes.** Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 217–218 (1949). (Russian)

The formula $T=2\pi(R/g)^{\frac{1}{2}}$ derived by M. Schuler [Phys. Z. **24**, 344–350 (1923)] gives a period of 84 minutes for a certain oscillation of a pendulum or gyroscope ($R$ is the radius of the earth). This author shows that the same formula also applies to the period of a disturbance of one gyroscope of a coupled system. *R. E. Gaskell.*

**Colombo, Giuseppe. Sopra le piccole oscillazioni di una superficie conica pesante.** Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **106**, 172–179 (1948).

The author derives an intrinsic differential equation of motion for the small oscillations, about a configuration of equilibrium, of a heavy inextensible membrane which is bounded by two fixed straight line segments $OA$ and $OC$, and by a variable arc $ABC$. It is assumed that the oscillations are such that the membrane is always a portion of a (variable) conical surface. *L. A. MacColl.*

**Colombo, G. Sull'equazione differenziale da cui dipendono le piccole oscillazioni di una superficie conica pesante.** Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **106**, 180–183 (1948).

In this note, which is a continuation of the one reviewed above, the author discusses the reality and signs of the eigenvalues of the problem, and also certain questions relating to the calculation of solutions of the differential equation. *L. A. MacColl* (New York, N. Y.).

**Braunbek, Werner. Eigenschwingungen endlicher Punktketten.** Ann. Physik (6) **4**, 169–173 (1948).

The author solves the system of differential-difference equations, linear with constant coefficients, for a symmetric linear lattice. *P. Franklin* (Cambridge, Mass.).

**Hostinský, Bohuslav. Influence des chocs transversaux sur le mouvement vibratoire d'une corde.** Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **45** (1944), 317–323 (1945).

The Lagrangian model consisting of $m$ similar mass points equidistantly disposed along a taut string is discussed under the standard assumptions of small oscillation theory for the following cases: (I) $m$ arbitrary, system unexcited; (II) $m=1$, system excited by impulses delivered to the $k$th mass point (here $k=m-1$) at arbitrary times $t_1, t_2, \cdots, t_i$ by a mass point incident at set velocity along a path perpendicular to the string; (III) $m=1$, system excited as in (II); however, the collisions occurring at regular intervals chosen at $\frac{1}{4}$, $\frac{1}{2}$, or the full natural period of the string; (IV) $m$ arbitrary, excitation as in (II).

Formulae obtained express: for case (I), the displacement and energy of the system as summation over fundamental modes; for case (II), recurrence relations for the displacement, velocity, velocity increment and energy of the individual modes, and total system at the $n$th impulse; for case (III), the limiting energy as $n$ becomes infinite during each chosen periodic excitation; for case (IV), recurrence relations for the displacement, velocity, velocity increment, and energy of the individual modes and total system at the $n$th impulse; also for the energy of the system at the $n$th collision of any fixed periodicity.

The calculations are straightforward, employing normal coordinates and summation over fundamental modes at each stage of the excitation process. The author makes use of the Volterra-Fredholm technique of decomposing a nonhomogeneous polynomial of degree $j-1$ into a sum of homogeneous polynomials of degree $1, 2, \cdots, j-1$ each having a determinant with roots giving the proper values. The model is incompletely described, there being no explicit statement concerning the removal of the incident foreign mass after each collision. Practical applications and effectiveness of the formulae in numerical calculations are not discussed. There are minor misprints. *J. H. Bigelow.*

**Hostinský, Bohuslav. The influence of transverse impulses on the oscillation of a string.** Rozpravy II. Třídy České Akad. **54**, no. 24, 24 pp. (1944). (Czech)
A summary is reviewed above.

**Buckley, R., and Whitfield, E. V. Impulses and constraints in classical mechanics.** Appl. Sci. Research A. **1**, 306–312 (1949).
The authors study the changes in velocity of a mechanical system due to impulses. They use tensor algebra and generalized Euclidean geometry to give geometrical interpretations of a number of principles, methods, and results.
*P. Franklin* (Cambridge, Mass.).

**Pars, L. A. An elementary proof of Stäckel's theorem.** Amer. Math. Monthly **56**, 394–396 (1949).

**Cotton, Émile. Sur certains rapprochements entre la géométrie des espaces de Riemann et la mécanique rationnelle classique.** Bull. Soc. Math. France **76**, 1–19 (1948).
The author deals with the well-known representation of the motion of a holonomic dynamical system with coordinates $q^i$ $(i=1, 2, \cdots, n)$ as the motion of a point in a Riemannian space of $n$ dimensions, pointing out the analogy between acceleration and tensor differentiation of vectors. Using a system of "quasi-coordinates" $\omega^i=\sum_j \Gamma_j^i dq^j$, he develops the formulae for tensor derivatives of vectors (and the equations of motion of a dynamical system) in these coordinates. He shows that the results can be easily extended to nonholonomic systems. *A. J. McConnell.*

**Castoldi, Luigi. Forze d'inerzia nei sistemi lagrangiani e loro carattere conservativo in taluni casi particolari.** Pont. Acad. Sci. Acta **11**, 63–69 (1947).
These remarks on the so-called inertial forces in Lagrangian systems center attention on a simple condition for conservativity and (in the case of a single particle) the well-known connection with forces of Lorentz type.
*D. C. Lewis* (Baltimore, Md.).

**Melchior, Paul. Sur la dynamique des solides. II.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 779–784 (1948).
[For part I see the same vol., 445–448 (1948); these Rev. **10**, 159.] The Lagrangian equations for the free motion of a rigid body are written down in terms of the three coordinates of the center of gravity and the nine direction cosines of a system of axes fixed in the body (with respect to a system fixed in space). Thus the number of generalized coordinates is $3+9=12$ instead of the usual 6 for such a problem, but there are, of course, six equations of constraint (holonomic) to be handled with Lagrangian multipliers in a classical manner. Analogous theories for the corresponding Hamiltonian and the Jacobi equations are also presented.
*D. C. Lewis* (Baltimore, Md.).

**Melchior, Paul. Dynamique d'un solide à liaisons non holonomes d'après la méthode variationnelle de Th. De Donder.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 785–788 (1948).
The variational principle of de Donder is applied to the problem of the rolling hoop. This problem, by the introduction of superfluous coordinates and the use of Lagrangian multipliers, is shown apparently to be capable of being written in Lagrangian form, even though it is nonholonomic. This state of affairs is not true in general, and the results here are not entirely devoid of suspicion. In the first place the use of Lagrangian multipliers for nonholonomic constraints is somewhat different from the corresponding use in the problem of Lagrange of the calculus of variations [as apparently employed by the author]. In the second place, the author, in reconciling his equations with those published in Appell's Traité de Mécanique Rationelle, treats the point of contact with the ground as if it were fixed with respect to axes fixed in the hoop [a manifest absurdity].
*D. C. Lewis* (Baltimore, Md.).

**De Donder, Th. Nouveau principe variationnel de la dynamique des solides à liaisons non holonomes de roulement.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 701–702 (1948).
A slight and somewhat obscure generalization, without details, of the paper of P. Melchior [see the preceding review], who treated the case of the rolling hoop. The author apparently means to extend the theory to any rigid body rolling on a plane. *D. C. Lewis* (Baltimore, Md.).

**De Donder, Th., et Melchior, Paul. Le principe de moindre contrainte de Gauss appliqué à la dynamique des corps solides à liaisons non holonomes.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 966–968 (1948).
The equations of motion of the rolling hoop are derived by application of Gauss's principle of least constraint.
*D. C. Lewis* (Baltimore, Md.).

**Aržanyh, I. S. Nonholonomic dynamical systems possessing a kinetic potential.** Doklady Akad. Nauk SSSR (N.S.) **65**, 809–811 (1949). (Russian)
Necessary and sufficient conditions under which the equations of a nonholonomic system possessing a potential can be presented in Lagrangean form are given, using tensor analysis, as integrability conditions of a system whose coefficients are expressed in terms of the explicit equations of motion. There are some misprints. *G. Y. Rainich.*

**Manarini, Mario. Alcune notevoli proprietà del centro d'inerzia relative alle rotazioni permanenti nella dinamica del corpo rigido con un punto fisso.** Boll. Un. Mat. Ital. (3) **3**, 214–219 (1948).
The author has given contributions to the properties of the Staude-Van der Woude motions of a rigid body about a fixed point under gravity [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **14**, 572–577 (1931)]; these motions are permanent rotations about a vertical axis and this axis is a principal axis of inertia of the body. He now gives a simpler and almost elementary treatment of the problem with some remarks on the Poinsot motion.
*O. Bottema* (Delft).

**Rankin, R. A. The mathematical theory of the motion of rotated and unrotated rockets.** Philos. Trans. Roy. Soc. London. Ser. A. **241**, 457–585 (1949).
The author makes a thorough and comprehensive study of the motion of a rocket during burning. He states that this paper incorporates earlier unpublished work of W. R. Cook, A. T. Wadley, and F. E. Mercer on fin stabilized rockets and work of C. L. Barham (rotating rockets) and E. T. J. Davies and H. G. Haden (wind effects). The problem is to devise a mathematical theory which will, after experimental measurement of suitable constants, predict position, velocity, angular position, and angular velocity of a rocket at the end of burning from a knowledge of these and other physical data at the beginning of burning. Of

necessity, since the work is intended for direct application to ballistic calculations, considerable detail is given, and this, in turn, requires a truly formidable list of notations.

Of the various assumptions made, the most important are the following. (a) The propellant is solid. (b) The burning time is short. In particular, the angular deviation of the trajectory during burning is small; this permits use of Siacci-Hitchcock-Kent approximations. (c) The rocket combustion does not affect the aerodynamic forces other than drag. (d) There is an axis of aerodynamic symmetry, i.e., an axis $A$ fixed in the rocket, such that if the velocity and angular velocity vectors are parallel to $A$ then so also are the aerodynamic force and torque. Relative to this axis, the aerodynamic force system is that of R. G. Fowler, E. G. Gallup, C. N. H. Lock, and H. W. Richmond [Philos. Trans. Roy. Soc. London. Ser. A. **221**, 295–387 (1920)], as amended by K. L. Nielsen and J. L. Synge [Quart. Appl. Math. **4**, 201–226 (1946); these Rev. **8**, 100]. (e) The various asymmetries are small, i.e., there is a small angle between the aerodynamic axis and the axis of minimum inertia, a small malalignment of the thrust and axial torque, a small malalignment of the center of mass of the propellant, etc.

The author derives the equations of motion under quite general conditions. Explicit solutions of these equations in terms of Fresnel integrals and quadratures of Fresnel integrals are given in the following cases. (1a) The spin is proportional to the velocity and (1b) there are no damping or Magnus forces or torques; (2) the acceleration is constant and (1b); or (3) certain special spin and stability conditions hold.

[It is to be noted that, under the assumption of constant acceleration and a restricted aerodynamic force system, solutions of essentially the same sort have been obtained by (not chronologically) J. B. Rosser, R. R. Newton and G. L. Gross [Mathematical Theory of Rocket Flight, McGraw-Hill, New York, 1947; these Rev. **9**, 108], C. H. Dowker, G. P. Hochschild, L. Blitzer and E. Davis, and A. S. Peters for fin stabilized rockets, by E. Davis and by A. S. Peters in the spin stabilized case. American air-to-ground firing tables are based on Peters' theory and tables. Except for the first reference, these have not been published.]

The author obtains an approximate solution for the equations of motion under the assumptions that the various coefficients are slowly varying functions of time. The method is similar to that of H. Jeffreys [Proc. London Math. Soc. (2) **23**, 428–436 (1924)], and to that of E. J. McShane, F. V. Reno, and the reviewer in a forthcoming book. The solution obtained is used in determining stability conditions. Two stability conditions are given which are analogous to those which arise in free flight. The third condition given is new in published work on exterior ballistics and indicates that instability of a resonance type may occur with a spinning fin stabilized rocket whose aerodynamic axis does not coincide with a principal axis of inertia. The paper concludes with pertinent tables.

It appears to the reviewer that there are two questions which are inadequately covered by the author's otherwise very competent discussion. First, the assumption of aerodynamic symmetry is unrealistic, since on production items it is quite customary for the line of zero lift to differ appreciably from the line of zero moment. Second, the author's discussion of the effect of canted fins seems inadequate, due to too optimistic an interpretation of the qualification "small." *J. L. Kelley* (Berkeley, Calif.).

**Corner, J. The ballistic effects of bore resistance.** Quart. J. Mech. Appl. Math. **2**, 232–245 (1949).

This paper discusses qualitatively, and presents a method for handling numerically, one of the recognized secondary problems of interior ballistics, assuming the differential equations of unresisted motion in their classical form. The author shows how to construct, for an arbitrary resistance-travel curve, weighting factors accounting for first order effects for nonvanishing variable axial resistance, handling separately effects prior to completed burning of the powder, and effects under adiabatic expansion of the gas thereafter, up to muzzle position of the projectile. Numerical comparison with known data shows that, at least in certain typical cases, the method provides satisfactory estimates even although the problem is theoretically far from linear. Numerical integration of the product of weighting factor and resistance yields the effect of bore resistance with any desired dependence upon travel. The effect on peak pressure, mentioned as much less important than that on muzzle velocity, is examined. Whether the numerous relatively simple assumptions traditionally made concerning chemical reactions, rapid mechanical stresses, motion of gas and of powder particles, pressure waves in gas and in the barrel, the throttling effects due to difference in diameter of powder chamber and bore, heat transfer, the flow of gas past the projectile, etc., even approximate the facts is not touched upon. "Though the results of the present paper form only an interim stage, it is possible that for a long time this may be all that is justified by our knowledge of bore resistance."

*A. A. Bennett* (Providence, R. I.).

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### *Hydrodynamics, Aerodynamics, Acoustics*

**Szegö, G. The virtual mass of nearly spherical solids.** Duke Math. J. **16**, 209–223 (1949).

Considérons un solide mobile $S$ dans un fluide (remplissant tout l'espace restant) incompressible, sans frottement, irrotationnel, au repos à l'infini. Si $S$ a un mouvement de translation avec vitesse de grandeur $U$, soit $T$ l'énergie du fluide. On appelle $W=T/\frac{1}{2}U^2$ la masse virtuelle de $S$ et on étudie spécialement sa moyenne $W_m$ pour toutes les directions de la vitesse. Après Pólya qui s'en occupa pour les ellipsoïdes et les cylindres indéfinis [Proc. Nat. Acad. Sci. U. S. A. **33**, 218–221 (1947); ces Rev. **9**, 111], l'auteur fait la comparaison avec une sphère de même volume. Il ne considère que le corps presque sphérique limité par une surface $r=1+\delta f(\theta, \varphi)$, où $\delta$ sera assez petit. Il donne, en négligeant les termes d'ordre supérieur à $\delta^2$, une expression de $W_m$, qui montre l'existence (présumée par Pólya) d'un minimum atteint pour la sphère seule. Il compose aussi les rayons des sphères qui ont respectivement même volume et même $W_m$ que $S$. Il complète par des considérations similaires sur des invariants déduits de $W$ analogues à $W_m$ et par une nouvelle proprieté du cas cylindrique.

*M. Brelot* (Grenoble).

**Fromm, Hans. Laminare Strömung Newtonscher und Maxwellscher Flüssigkeiten.** Z. Angew. Math. Mech. **28**, 43–54 (1948).

The author considers flow of fluids satisfying the Maxwell type of stress-strain relations; i.e., the rate of deformation is expressible as a linear combination of the stress and the rate of change of stress (following the motion of an element

of fluid). In particular, the stationary flows between two parallel plates and through a circular tube are considered. Two possible solutions are obtained for the distribution of velocity and pressure, one "stable" and the other "unstable." The "stable" solution is similar to the solution for the Newtonian fluid and approaches it in the limit of vanishing elasticity. The "unstable" solution has an infinite velocity along the axis of symmetry. Pressure measurements and the relation of the present theory to structural viscosity of colloidal solutions are discussed. *C. C. Lin.*

**Hinze, J. O. Forced deformations of viscous liquid globules.** Appl. Sci. Research A. 1, 263–272 (1949).

The effects of viscosity consistent with slight deformations of a spherical liquid globule are obtained, in the cases of small and great viscosity, by solving the linearised hydrodynamical equations subject to the conditions of a prescribed external axisymmetrical pressure distribution, and zero tangential stress at the boundary. The results differ from those given by Lamb [Hydrodynamics, 6th ed., Cambridge University Press, 1932, § 355] which were obtained from energy considerations assuming irrotational flow within the globule but not linearised equations.

*L. M. Milne-Thomson* (Greenwich).

**Dean, W. R., and Montagnon, P. E. On the steady motion of viscous liquid in a corner.** Proc. Cambridge Philos. Soc. 45, 389–394 (1949).

The author considers a two-dimensional flow of a viscous fluid in a corner formed by two rigid straight boundaries inclined at an angle $\alpha$. In the neighborhood of the corner, the motion is slow enough to neglect the inertia term and the stream-function $\psi$ is biharmonic. The simplest biharmonic function conforming to the non-slip conditions is

$$\psi=[A\cos(n+1)\vartheta+B\sin(n+1)\vartheta+C\cos(n-1)\vartheta+D\sin(n-1)\vartheta]r^{n+1},$$

where $r$, $\vartheta$ are plane polar coordinates and $A$, $B$, $C$, $D$ and $n$ are constants. It follows, from the boundary conditions, that the parameter $n$, for each $\alpha$, satisfies

$$(n\alpha)^{-1}\sin(n\alpha)=\pm\alpha^{-1}\sin\alpha.$$

It is found that when $\alpha$ lies in the range $180°\leqq\alpha\leqq360°$, $n$ must satisfy $1\geqq n\geqq\frac{1}{2}$; and when $\alpha$ lies in the range $146.3°\leqq\alpha\leqq180°$, $n$ must satisfy $1.76\geqq n\geqq1$. When $\alpha<146.3°$, $n$ is complex. The relation $n(\alpha)$ is plotted. (The real part is plotted when $n$ is complex.) The limiting case of small $\alpha$ is also examined and an asymptotic solution is found.

*Y. H. Kuo* (Ithaca, N. Y.).

**Lapwood, E. R. Convection of a fluid in a porous medium.** Proc. Cambridge Philos. Soc. 44, 508–521 (1948).

The author considers the instability of a layer of fluid subject to a vertical temperature gradient in a porous medium, and considers the influence of the resultant convective flow on the heat transfer across the layer. The resistance to fluid flow is assumed proportional to the velocity (Darcy's law), and consequently the condition of no slip at the boundaries is not imposed. It is shown that under certain conditions convective flow may occur. The criterion for marginal stability is obtained for three sets of boundary conditions, and the motion described. The correction to the usual method of calculation of heat flow is found to be not large. *C. C. Lin* (Cambridge, Mass.).

**Voščinin, A. P. The calculation of homogeneous earth dams constructed on a permeable foundation.** Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 761–768 (1948). (Russian)

This paper is essentially a reprint of a previous note [C. R. (Doklady) Acad. Sci. URSS (N.S.) 25, 735–737 (1939)] to which a numerical example has been appended. The filtration of water through the dam is studied following the method of Zhukovski, in which the complex potential $W=\varphi+i\psi$, $0\leqq\varphi\leqq H$, $0\leqq\psi\leqq q$, is subjected to a Schwarz-Christoffel transformation, leading to the final general solution $\operatorname{tn}[\frac{1}{2}\pi T^{-1}(Wi-z+\frac{1}{2}S)]=k\operatorname{sn}(-K+2KW/H,k)$, where $T$ is the depth of the lowest point of the permeable foundation, and $S$ is the horizontal length of the line of depression. *A. W. Boldyreff* (Albuquerque, N. M.).

**Kirkham, Don. Flow of ponded water into drain tubes in soil overlying an impervious layer.** Trans. Amer. Geophys. Union 30, 369–385 (1949).

**Verigin, N. N. The filtration of water from channels in an irrigation system.** Doklady Akad. Nauk SSSR (N.S.) 66, 589–592 (1949). (Russian)

**Verigin, N. N. On unstationary motion of ground water in the neighborhood of reservoirs.** Doklady Akad. Nauk SSSR (N.S.) 66, 1067–1070 (1949). (Russian)

*Coburn, N., and Dolph, C. L. **The method of characteristics in the three-dimensional stationary supersonic flow of a compressible gas.** Proc. Symposia Appl. Math., Vol. I, pp. 55–66. American Mathematical Society, New York, N. Y., 1949. $5.25.

This paper is an exposition and development of the three-dimensional method of characteristics following an attack similar to that of E. Titt [Ann. of Math. (2) 40, 862–891 (1939); these Rev. 1, 57]. The point of view is generally that of differential geometry and tensor notation is used consistently throughout. A curvilinear coordinate system is established using two families of characteristic surfaces plus a third family of surfaces whose members pass through the intersections of the characteristic surfaces. The equations of motion of a steady supersonic isentropic irrotational flow are expressed as 8 nonlinear first order partial differential equations for the space variables and the velocity components. For the particular case where the intersections of the characteristic surfaces lie in the equipotential surfaces the equations are simplified and conditions are found for solutions analogous to those for two-dimensional simple waves, two-dimensional and axially symmetric flows appearing as special cases. No practical examples are given.

*W. D. Hayes* (Providence, R. I.).

**Cap, Ferdinand. Über zwei Verfahren zur Lösung eindimensionaler instationärer gasdynamischer Probleme.** Acta Physica Austriaca 2, 224–238 (1949).

Two procedures are presented for calculating the unsteady one-dimensional motion of a gas under the assumption of isentropy. One of these is the method of characteristics of Riemann derived by means of the Legendre transformation. The other is the graphical method of F. Schultz-Grunow [Forschung Gebiete Ingenieurwesens. Ausg. B. 13, 125–134 (1942)], which is described and extended. The application of this method to nonadiabatic flow processes is indicated.

*W. D. Hayes* (Providence, R. I.).

**Luntz, Michel. Aérodynamique moléculaire. II.** Recherche Aéronautique **1949**, no. 8, 9–12 (1949).

Continuing his investigation of free molecular flows [Recherche Aéronautique **1949**, no. 7, 17–33 (1949); these Rev. **10**, 491], the author calculates the temperature and the polar, relation between lift and drag with angle of attack as parameter, of a polished aluminum flat wing with radiation equilibrium. The solar radiation is not considered, but the earth radiation is taken into account by assuming a constant earth temperature of 15°C. Two types of wing are considered: thermal insulation between the top and the bottom surfaces, and perfect conductivity between these surfaces. The atmosphere is assumed to have the same characteristics as that at ground. Results are presented in graphs for two Mach numbers 2.39 and 9.55.

*H. S. Tsien* (Pasadena, Calif.).

**Stanyukovič, K. P. Automodels of plane and axisymmetric steady motion of a gas.** Doklady Akad. Nauk SSSR (N.S.) **64**, 29–32 (1949). (Russian)

The author treats families of steady flows of a compressible fluid which fall under his automodel specification. A flow is termed "automodel" if the solution depends essentially upon a suitably chosen single independent parameter which is itself a function of the originally specified independent variables. In the cases investigated the restriction to plane or axi-symmetric flow is made at the outset. No a priori specification is made as to irrotationality or isentropy. Prandtl-Meyer flow and the conical flow of Busemann [Z. Angew. Math. Mech. **9**, 496–498 (1929)] are obtained as special cases. Some more general solutions are obtained in the form of unsolved differential equations. Finally, it is noted that for the most general case of steady spatial flows it is impossible to find automodel solutions which depend on only one parameter. *W. D. Hayes* (Providence, R. I.).

**Stanyukovič, K. P. Automodels of plane and axisymmetric unsteady motion of a gas.** Doklady Akad. Nauk SSSR (N.S.) **64**, 179–181 (1949). (Russian)

The author extends his automodel concept to unsteady plane and axi-symmetric flows of a compressible fluid. As in the steady case solutions of the assumed forms are obtained in the form of differential equations. As special cases the irrotational planar flows with point symmetry discussed by L. Sedov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 91–93 (1945); these Rev. **7**, 140] and by the author [ibid. **48**, 310–312 (1945); these Rev. **7**, 446] are found. *W. D. Hayes* (Providence, R. I.).

**Stanyukovič, K. P. Automodels of the motion of a gas in a gravitational field.** Doklady Akad. Nauk SSSR (N.S.) **64**, 467–470 (1949). (Russian)

The automodel type of similitude is applied to the unsteady motion of a gas with complete spherical symmetry under the influence of its own internal gravitational field. Several families of solutions are investigated, some of which may have astrophysical applications. In the stationary case an equation of the Emden type is obtained, which has, in addition to the known solution for $C_P/C_V=k=2$, a solution expressed as a quadrature for $k=6/5$. *W. D. Hayes.*

**Prim, R. C., III. A note on the substitution principle for steady gas flow.** J. Appl. Phys. **20**, 448–450 (1949).

Munk and Prim have shown [Proc. Nat. Acad. Sci. U. S. A. **33**, 137–141 (1947)] that for steady flow in the absence of extraneous force fields of a nonviscous, thermally nonconducting gas with constant specific heats, the so-called "substitution principle" is true, i.e., if in any flow field the densities are multiplied by $m$ and the velocities by $1/\sqrt{m}$, where $m$ is a positive function which is a constant along each individual streamline, the new flow field will be a possible flow of the same gas having the same pressure distribution and Mach number distribution as the original one. In this note, the author shows that this principle is generally true if the fluid has an equation of states of $\rho=P(p)S(s)$ where $\rho$ is density, $p$ pressure, and $s$ entropy. Furthermore, if $P(p)=p^k$, $k<1$, then the continuity equation can be reduced to one involving only the reduced velocity vector $\mathfrak{w}$, the velocity vector divided by the ultimate velocity for each streamline. The substitution principle reduces all flows which are nonisoenergetic in a direction normal to the streamlines to isoenergetic flows. Unfortunately, combustion problems generally do not fall into this class.

*H. S. Tsien* (Pasadena, Calif.).

***Opatowski, I. Two-dimensional compressible flows.** Proc. Symposia Appl. Math., Vol. I, pp. 87–93. American Mathematical Society, New York, N. Y., 1949. $5.25.

In the first part of the paper the author shows the existence of several classes of steady two-dimensional flows of a compressible gas. These flows are actually degenerate three-dimensional flows. The principal tools used in the investigation are (1) a classification by T. Levi-Civita [Mem. Accad. Sci. Torino (2) **49**, 105–152 (1899)] of degenerate metric tensors, and (2) a classification by the author of degenerate velocity vector fields in which the components obtained by projecting the velocity vector on the tangents to the various coordinate lines are degenerate. By an examination of the potential equation for irrotational flows, the author shows that if the velocity components are degenerate in the above sense, then cylindrical, axially symmetric, helical, conical and spiral flows are possible. The details of the calculations are omitted. Further, the author claims that his calculations show that the above types of flow are permissible rotational flows.

In the second part of the paper, the author examines the problem of determining the two independent integrals for the stream function when suitable degeneracies exist in the contravariant components of the velocity. By recognizing the fact that $\rho a^3$ ($\rho$=density, $a$=local sound speed) constitutes a last multiplier of Jacobi and by use of the classical Jacobi theory, the author obtains formulas for the two desired independent integrals in the cases of spherical, helical, conical and spiral flows.

Reviewer's note. It is not clear that, in the helical flows, the author is referring to the projected components of the velocity vector. There does remain the problem of examining degeneracies in the components of the velocity vector obtained by decomposing according to the parallelogram law. These latter components are as representative of the velocity vector as the projected components. That is, one set of components correspond to use of covariant base vectors, the other (projected) set of components correspond to use of the reciprocal contravariant base vectors.

*N. Coburn* (Ann Arbor, Mich.).

**Bergman, Stefan. Two-dimensional transonic flow patterns.** Amer. J. Math. **70**, 856–891 (1948).

The author obtains a general solution of the hodograph equations of plane steady homentropic gas flow, involving

an arbitrary function of a complex variable and valid in the subsonic region. He obtains another general solution involving two functions of a real variable and valid in the supersonic region. For the Tricomi equation, which is an approximation valid near the sonic speed, he is able to connect the solutions by analytic continuation. It is difficult to assess the work as a piece of mathematical physics until some applications of it have been described. Similar expressions have been published by other authors [e.g., Lighthill, Proc. Roy. Soc. London. Ser. A. **191**, 352–369 (1947); these Rev. **9**, 391], though the analysis of the Tricomi equation has not been previously extended so far. Their value in elucidating transonic gas flow has yet to be established. *M. J. Lighthill* (Manchester).

**Bergman, Stefan. Two-dimensional transonic flow patterns.** Harvard University, Graduate School of Engineering. Operator Methods in the Theory of Compressible Fluids. Work Performed under Contract NOrd 8555 Task F. Technical Report no. **10**. Cambridge, Mass., 1948. 47 pp.

The paper is identical with that reviewed above except for an appendix setting out at length some straightforward calculations of which only the results were quoted previously. *M. J. Lighthill* (Manchester).

***Bergman, S. On tables for the determination of transonic flow patterns.** Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 13–36. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

This paper is another description, with tables, of the author's method for finding steady two-dimensional irrotational transonic airflow patterns, as expounded previously in the papers reviewed above. *M. J. Lighthill.*

***Bers, Lipman. An existence theorem in two-dimensional gas dynamics.** Proc. Symposia Appl. Math., Vol. I, pp. 41–46. American Mathematical Society, New York, N. Y., 1949. $5.25.

The following existence theorem is proved. Let there be given a profile $P$ and a point $z_T$ on $P$. Let $P$ be convex and continuously curved, except, perhaps, for a sharp trailing edge at $z_T$. For every angle of attack and every value of the maximum local speed there exists a potential gas flow around $P$ which is uniform at infinity, satisfies the Chaplygin density speed relation, and possesses at $z_T$ a stagnation point. Here the Chaplygin relation referred to is that corresponding to $\gamma=-1$, where $\gamma$ is the ratio of specific heats. The proof is given for symmetrical flows, but it is stated that the extension to asymmetric cases can be made. The method used here is to relate the problem to the solution of a certain integral equation and to apply a theorem of Leray and Schauder [Ann. Sci. École Norm. Sup. (3) **51**, 45–78 (1934)]. It is also shown that potential flow of a Chaplygin gas past a convex corner is impossible. In a note added in proof, the author states that he can replace the condition that $P$ is convex by a less restrictive condition. *W. R. Sears* (Ithaca, N. Y.).

**Ehlers, F. E. Methods of linearization in compressible flow. II. Hodograph method.** Prepared under the supervision of G. F. Carrier. Tech. Rep. no. **F-TR-1180B-ND (GDAM A-9-M IV/II)**. Headquarters Air Materiel Command, Wright Field, Dayton, Ohio. vi+255 pp. (1948).

This is a faithful noncritical review of all works on two-dimensional steady compressible flow by the hodograph method. The only exception is some work on the transonic flow in a symmetrical channel, using the solutions of the Tricomi equation, by the author and G. F. Carrier. This result has the advantage over similar work by Lighthill [Proc. Roy. Soc. London. Ser. A. **191**, 323–341 (1947); these Rev. **9**, 391] in that the solution is in closed form and there is no difficulty in analytic continuation. In general, the exposition is detailed and no reference to the original papers is necessary. In an appendix, the previously tabulated functions for the hodograph method are listed and their notations compared as an aid to the research workers in this field. The reviewer wishes to point out particularly the account of work of Manwell [Philos. Mag. (7) **36**, 499–510 (1945); these Rev. **7**, 498] where the flow around a circular cylinder is solved by directly satisfying, in the hodograph plane, the physical boundary conditions. [However, the validity of Manwell's detailed calculations and results is questionable as he does not consider the analytic continuation of the solution near the origin in the hodograph plane beyond its region of convergence.] *H. S. Tsien.*

**Reissner, Eric. Note on the theory of lifting surfaces.** Proc. Nat. Acad. Sci. U. S. A. **35**, 208–215 (1949).

The paper begins with the integral equation of a thin lifting surface, for a rectangular wing in incompressible inviscid flow. It is shown how the Prandtl lifting-line approximation for the lift distribution is obtained by neglecting a term in one integral, and how the analogous equation of Weissinger's "$F$-method" [Zentrale für Wissenschaftliches Berichtswesen des Generalluftzeugmeisters, Forschungsber., no. **1553** (1942); translated in Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1120** (1947); these Rev. **8**, 542] is obtained by making less drastic approximations to this same term. Finally Weissinger's "$L$-method" results from a further approximation, which, the author points out, is only consistent with the previous ones.

The author now proposes a new procedure, which involves the integral equations for both lift and moment distributions, which are obtained from the original integral equation by integrating across the chord according to Munk's well-known formulas of plane airfoil theory. The assumptions made here in reducing the troublesome integral are less restrictive than in Weissinger's work, and the new equations are applicable to problems outside the scope of a lifting-line procedure. An example is given. It is mentioned that the method could be extended to an $n$-equation procedure, by forming $n$ integrals of Munk's type. *W. R. Sears.*

**Heaslet, Max. A., and Lomax, Harvard. The application of Green's theorem to the solution of boundary-value problems in linearized supersonic wing theory.** Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1767**, 41 pp. (1949).

Problems in the linearised theory of three-dimensional steady, and two-dimensional unsteady, high-speed flow past wings, are treated by use of formulae (deducible from the work of Hadamard) which express a solution of the wave equation in terms of its values on a time-like plane, in cases when the value ahead of some space-like plane is known to be zero. For determining pressure from shape with a flat lifting surface, or for determining shape from pressure with a symmetrical wing at zero incidence, the treatment involves the solution of an integral equation. However, in the problems treated, this solution is possible by inversion of the Poisson integral relating allied Fourier series. The pressure

distribution and downwash are calculated for supersonic flow past a flat triangular wing with subsonic leading edges. The change in pressure distribution, over an aerofoil flying at any constant Mach number other than one, as a result of an abrupt change of incidence, is plotted against time. Finally a special study is made of the steady flow at Mach number one. The theory is simplified in this case by the disappearance of one of the second derivatives from the linearised equation. It appears, however, that, at least for certain boundaries, the deduced velocity field still satisfies the assumptions governing the linearisation of the equations.

*M. J. Lighthill* (Manchester).

***Kuo, Y. H. On the stability of transonic flows.** Proc. Symposia Appl. Math., Vol. I, pp. 72–73. American Mathematical Society, New York, N. Y., 1949. $5.25.

This is a preliminary report of an investigation of the stability of two-dimensional compressible potential flow involving an imbedded supersonic region. A flow past a thin body is assumed, for which the velocity potential is expanded in a power series in the thickness parameter. The time-dependent disturbance assumed is a triangular pulse, moving upstream, its head consisting of a weak shock wave. It is proposed to trace the history of this pulse, in terms of the velocity of its headwave and the slope of its tail.

*W. R. Sears* (Ithaca, N. Y.).

**Craggs, J. W. The application of the hodograph method to problems of subsonic compressible flow in two dimensions.** Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2273 (9801), 16 pp. (1949).

The author gives a simple and clear exposition of the Kármán-Tsien method and its extension to flow around a cylinder with circulation by C. C. Lin [Quart. Appl. Math. 4, 291–297 (1946); these Rev. 8, 418], A. Gilbarg [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1170 (1947); these Rev. 8, 417], L. Bers [ibid., no. 969 (1945); these Rev. 7, 497], P. Germain [C. R. Acad. Sci. Paris 223, 532–534 (1946); these Rev. 8, 237] and K. v. Jaechel [unpublished]. The author shows explicitly that the Kármán-Tsien method reduces to the Janzen-Rayleigh method when terms in $M_\infty^4$ (Mach number at infinity) are neglected and to the Prandtl-Glauert method when the disturbances to the free stream are small. The profile distortions for a particular symmetrical Joukowsky airfoil of 13% thickness are calculated for $M_\infty=0.6$ at zero incidence and at angle of attack equal to 0.05 radian.

*H. S. Tsien* (Pasadena, Calif.).

**Hönl, H. Two-dimensional wing theory in the supersonic range.** Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1238, 34 pp. (1949).

[Translated from Zentrale für Wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters (ZWB), Forschungsber., no. 1903 (1944).] Using the method of the acceleration potential, the author sets up the integral equation for the doublet distribution representing a thin, harmonically oscillating, two-dimensional airfoil in supersonic flow. His method of solution is based on the integral properties of the $\delta$-function, and leads to results identical with those of L. Schwarz [reports abstracted in Karp, Shu, and Weil, Tech. Rep. no. F-T-1167-ND (GDAM A-9-M III), Headquarters Air Materiel Command, Wright Field, Dayton, Ohio (1947); these Rev. 9, 632].

*W. R. Sears* (Ithaca, N. Y.).

**Golubev, V. V. On the theory of a wing of small span.** Učenye Zapiski Moskov. Gos. Univ. Mehanika 122, tom II, 3–16 (1948). (Russian)

It is well known that for aspect ratios less than about unity the agreement between experimental and theoretical lift distribution based on the lifting line theory of Prandtl breaks down. The paper under review presents a modification of the conventional theory by assuming that the free vortex sheet does not leave the wing at the trailing edge but at some line of the suction side at a distance approximately one quarter of the chord length from the leading edge and in the same plane as the main stream. The wing itself is supposed to be replaced by a plane. The first assumption leads to a secondary stream leaving the wing at the tips. The secondary stream, contrary to the main stream, is not supposed to be subjected to Joukowski's hypothesis. The presence of the secondary stream, in the case of elliptic loading, produces an increase of circulation proportional to the square of the angle of attack $\theta$ and

$$(1)\qquad \varphi(\chi,\theta)=\frac{2\lambda}{\pi+2\lambda}\left[1+\pi\sqrt{3}\frac{\theta}{\pi+2\lambda}\right]$$

gives the increase of the lift $L$ as compared with a two-dimensional motion for given aspect ratio $\lambda$ and angle of attack $\theta$. Instead of a linear relation as given by the lifting line theory, a quadratic relation between the lift coefficient $L_a$ and the angle of attack $\theta$ is set up. The main results of the paper may be summed up as follows. (i) The values of $L_a$ calculated by the conventional theory are too small; a better approximation of the observed values is given by formula (1). (ii) For very small aspect ratios ($\lambda<1$), the critical angle of attack increases rapidly. For $\frac{1}{2}<\lambda<1$, the measured values for the critical angle of attack are higher than the calculated ones [cf. Golubev, Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] 1947, 261–270].

*E. Leimanis* (Vancouver, B. C.).

**Carrière, Pierre. Écoulements supersoniques infiniment voisins.** C. R. Acad. Sci. Paris 228, 1632–1634 (1949).

The author formulates the problem of finding a supersonic potential flow slightly disturbed from a given supersonic potential flow. The main simplification comes from the linearization of the equation and thus the integration can be carried out with fixed characteristics.

*H. S. Tsien* (Pasadena, Calif.).

**Ferrari, Carlo. Interference between wing and body at supersonic speeds. Analysis by the method of characteristics.** J. Aeronaut. Sci. 16, 411–434 (1949).

**Rott, N. Oscillating airfoils at Mach number 1.** J. Aeronaut. Sci. 16, 380–381 (1949).

Beginning with the velocity potential for a harmonically oscillating point source of sound in a stream of sonic velocity, the author computes the potential for an oscillating two-dimensional airfoil. If $\nu$ is the angular frequency and $k$ denotes $\nu/a$, $a$ being the speed of sound, the calculated pressures on the airfoil behave like $k^{-\frac{1}{2}}$ as $k\to 0$, but are finite for $k>0$. As examples, the plunging and pitching of a flat airfoil are considered, and the lift and moment are calculated. [Reviewer's note: the limitations on the validity of this type of linearized theory near sonic speed were examined by Lin, Reissner, and Tsien, J. Math. Physics 27, 220–231 (1948); these Rev. 10, 162].

*W. R. Sears.*

**Sagomonyan, A. Ya. Operational methods in gas dynamics.** Vestnik Moskov. Univ. 3, no. 5, 53–58 (1948). (Russian)

The operational calculus is incorrectly applied to the problem of an oscillating deformable obstacle in a supersonic flow. *G. F. Carrier* (Providence, R. I.).

**Miles, John W. On harmonic motion at supersonic speeds.** J. Aeronaut. Sci. 16, 378–379 (1949).

Assuming harmonic time dependence, the linear-perturbation equation can be written $\psi_{yy}+\psi_{zz}=\beta^2\psi_{xx}+k^2\psi$, where the perturbation velocity potential is $\psi \exp[i(\omega t-\lambda x)]$, $\beta^2$ denotes $M^2-1$, $M$ is the stream Mach number, $\omega$ is the angular frequency, $\lambda=Mk/\beta$, and $k=\omega/\beta c$. Here $c$ denotes the sound speed, $t$ the time, and $x, y, z$ the usual coordinates, $x$ being measured in the stream direction. The author proposes to neglect $O(k^2)$ for slow oscillations; then steady-flow solutions can be employed. As an example, he calculates the moment on a flat two-dimensional airfoil executing pitching oscillations, using the appropriate steady-flow Green's function, and obtains a result consistent with his previous work [e.g., same J. 14, 351–358 (1947); these Rev. 8, 510]. He also discusses the application of this approximate method to the rectangular wing tip and the slender body of revolution. *W. R. Sears* (Ithaca, N. Y.).

**Miles, John W. On the oscillating rectangular airfoil at supersonic speeds.** J. Aeronaut. Sci. 16, 381 (1949).

This is a short note presenting the results of an unpublished analysis. It states that the pressure distribution on the tip of a rectangular wing carrying out arbitrarily distributed sinusoidal oscillations has been calculated in the linear-perturbation approximation. The final results are given here for the special case where the downwash is constant over a spanwise strip. This result differs from the two-dimensional result only by the appearance of an extra term. For vanishingly slow oscillations the known steady-flow formula is obtained. More detailed results are to be published elsewhere. *W. R. Sears* (Ithaca, N. Y.).

**Miles, John W. A note on a solution to Possio's integral equation for an oscillating airfoil in subsonic flow.** Quart. Appl. Math. 7, 213–216 (1949).

The equation is

$$w(x,t)=(1/2\pi\mu)\int_{-1}^{1}\gamma(\xi,t)\exp[ik(\xi-x)](x-\xi)^{-1}Rd\xi,$$

where $R$ denotes $R[M,\mu^2k(x-\xi)]$, defined by

$$R(M,y)=\tfrac{1}{2}i\pi My\int_{-\infty}^{y}\exp(iu)H_1^{(2)}(M|u|)\,|u|^{-1}du,$$

where $H_1^{(2)}(z)$ is the Hankel function. Here $w(x,t)$ is the downwash along the chord of a thin oscillating wing (in this case proportional to $\exp(i\omega t)$), $\gamma(x,t)$ is the pressure, $M$ is the Mach number, $\mu$ denotes $(1-M^2)^{\frac{1}{2}}$, and $k$ is proportional to $\omega$. [See Küssner, Luftfahrtforschung 17, 370–378 (1940); these Rev. 2, 330.] The method used here is to expand the Hankel function in powers of $M^2$ (including terms in $M^{2n}\log M$), and finally to use the technique of solution employed by Schwarz in the incompressible case [ibid., 379–386 (1940); these Rev. 3, 286]. The author points out how this method can be improved by iteration, and carries out the calculation for the terms of order $M^2$, $M^2\log M$.

*W. R. Sears* (Ithaca, N. Y.).

**Miles, John W. Errata—aerodynamic forces on an oscillating flap at supersonic speeds.** J. Aeronaut. Sci. 16, 442–443 (1949).

The paper appeared in the same J. 15, 565–568 (1948); these Rev. 10, 163.

**Djang, Gwoh-Fan. Solution of Prandtl's boundary layer equation by a modified iteration method.** Acad. Sinica Science Record 2, 155–157 (1948).

This note gives an iteration process for the boundary layer equation by calculating the shear from the velocity by the momentum relation and the velocity from shear by Newton's law. A satisfactory result is obtained for the Blasius case. *C. C. Lin* (Cambridge, Mass.).

**Görtler, H. Grenzschichtentstehung an Zylindern bei Anfahrt aus der Ruhe.** Arch. Math. 1, 138–147 (1948).

The development of a laminar boundary layer on a two dimensional body started from rest, either impulsively or with uniform acceleration, was discussed by Blasius [Z. Math. Phys. 56, 1–37 (1908)] who obtained by an iteration procedure the solution as a series of ascending integral powers of the time $t$ elapsed after the motion was initiated. Blasius calculated the first two terms of the series explicitly and the third term has been obtained by Goldstein and Rosenhead [Proc. Cambridge Philos. Soc. 32, 392–401 (1936)] for the impulsive start. In particular, the time for first separation and position of separation may be approximated. The case of the boundary layer on an ellipse, symmetric with respect to the free stream, was discussed by Tollmien [Handbuch der Experimentalphysik, v. $4_1$, pp. 275–279, Akademische Verlagsgesellschaft, Leipzig, 1931] who first pointed out the fact that when the ratio of the axis normal to the stream to that parallel to the stream exceeds $\sqrt{\frac{1}{2}}$, the separation no longer takes place at the rear stagnation point but rather at each of two points located symmetrically with respect to this point.

In the present paper the author reviews this work and extends the approximation for the ellipse beyond that used by Tollmien [loc. cit.] with the result that the second and third terms in the series have no effect on the critical ratio of the ellipse's axes. He then obtains approximate values for the distance travelled by the ellipse before separation, according to the second approximation, for all ellipse-axis ratios and for impulsive as well as uniform acceleration.

*F. E. Marble* (Pasadena, Calif.).

**Scholkemeier, Friedrich-Wilhelm. Lösung der Prandtlschen Grenzschichtdifferentialgleichungen mit Hilfe von Potenzreihenentwicklungen.** Arch. Math. 1, 270–277 (1949).

The author obtains series solutions of the equations of axi-symmetrical boundary layer flow, in powers of the distance $x$ along a meridian section of the solid surface, thus extending to the axi-symmetrical case the work of Howarth [Proc. Roy. Soc. London. Ser. A. 164, 547–579 (1938)]. The coefficients are fully tabulated. *M. J. Lighthill.*

**Young, A. D. Skin friction in the laminar boundary layer in compressible flow.** Coll. Aeronaut. Cranfield. Rep. no. 20, 28 pp. (3 plates) (1948).

Hantzsche and Wendt [Jahrbuch 1942 der Deutschen Luftfahrtforschung, I40–I50; these Rev. 9, 314], Crocco [Monografie Scientifiche di Aeronautica, no. 3 (1946)= Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 187 (1947); these Rev. 10, 75] and many others have discussed

the laminar boundary layer due to the flow of a compressible fluid along a semi-infinite flat plate without pressure gradient. When no heat is transferred between the fluid and the plate, Hantzsche and Wendt have expressed the skin friction coefficient $C_f$ in the form

$$C_f\sqrt{R_x}=G(0)/[1+\tfrac{1}{2}(\gamma-1)M_1^2\sigma^{\frac{1}{2}}]^{\frac{1}{2}(1-\omega)},$$

where $R_x$ is the Reynolds number based on the distance from the leading edge of the plate, $M_1$ the free stream Mach number, $\sigma$ the Prandtl number, $\omega$ the exponent characteristic of the variation of viscosity with temperature ($\omega\approx 1$) and $G(0)$ is a function of $\omega$, $\sigma$ and $M_1$ which, according to numerical evaluation, varies only slightly over the useful ranges of these parameters. Using this result as a rational basis for the form of his expression, the author develops an empirical expression for $C_f$ by fitting the graphical results of Hantzsche and Wendt with an appropriate formula for $G(0)$. Then, by use of Crocco's general results, the expression is extended to include the case of heat transfer. The author states that the constants in the expressions

$$C_f\sqrt{R_x}=0.664[0.45+0.55i(0)+0.09(\gamma-1)M_1^2\sigma^{\frac{1}{2}}]^{\frac{1}{2}(\omega-1)},$$

for an enthalpy distribution at the surface of $i(0)$ times the free stream enthalpy and

$$C_f\sqrt{R_x}=0.664[1+0.365(\gamma-1)M_1^2\sigma^{\frac{1}{2}}]^{\frac{1}{2}(\omega-1)}$$

for the insulated plate, have been adjusted to give an accuracy of 1% up to a Mach number of 10.

With these simple analytic expressions for the skin friction, the author discusses the cooling and heating of flat plates as affected by dissipation in the fluid and also considers the boundary layer on an infinite cylinder. Finally the author employs his result in estimating the point of boundary layer separation in the manner of Pohlhausen, as well as its variation with free stream Mach number. The value of this final calculation seems doubtful to the reviewer. *F. E. Marble* (Pasadena, Calif.).

**Jones, C. W. On a solution of the laminar boundary-layer equation near a position of separation.** Quart. J. Mech. Appl. Math. **1**, 385–407 (1948).

The author discusses a solution of the laminar boundary-layer equations recently given by Goldstein [Quart. J. Mech. Appl. Math. **1**, 43–69 (1948); these Rev. **10**, 270]. Conditions near an assumed singularity at a point of separation are investigated numerically. The result shows that Goldstein's solution is very likely to be valid, although a formal proof appears to be difficult. In addition, an improved form of the "outer" solution of von Kármán and Millikan is described in the particular case of a linear pressure gradient. *C. C. Lin* (Cambridge, Mass.).

**Faĭnzil'ber, A. M. Some cases of the reduction of the equations of motion in the boundary layer of a viscous compressible fluid to ordinary differential equations.** Doklady Akad. Nauk SSSR (N.S.) **64**, 775–778 (1949). (Russian)

The boundary layer flow of a compressible viscous fluid is considered. A transformation of variables is made so that for several different pressure distributions one can reduce the usual equations to an ordinary differential equation. These pressure distributions are not tied in with any definite physical problem. The author fails to state that he has tacitly assumed a thermally insulated boundary and a fluid of Prandtl number unity. Certain errors occur in the analysis which invalidate the final equations given, but these are easily corrected. *G. F. Carrier* (Providence, R. I.).

**Kaufmann, Walter. Über die Aufwickelung einer instabilen Wirbelschicht von endlicher Breite.** S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1945/46**, 109–130 (1947).

This paper extends Betz's theory of the development of the vortex sheet trailing behind a wing. It is assumed that there are two end vortex tubes of circular cross-section with a sheet stretched in between. A plane of symmetry perpendicular to the sheet divides it into two parts with vorticity of opposite signs. The centroid of each half is at a fixed distance $a$ from this plane (following Betz). Then it is shown that the radius $r$ of the vortex tube is given by $r^2=\frac{1}{2}\{f(x)-a^2+a(a^2+2f(x))^{\frac{1}{2}}\}$, after it has collected all the vortex lines which were originally (i.e., at the wing) at a distance $x$ or more from the plane of symmetry. Here $f(x)$ is given in terms of the circulation distribution $\Gamma(x)$ at the wing by the relation $[f(x)]^{\frac{1}{2}}=\{\Gamma(x)\}^{-1}\int_x^{b/2}\Gamma(\xi)d\xi$, where $b$ is the span of the wing. The location of the vortex and the pressure distribution inside the vortex system are also worked out. *C. C. Lin* (Cambridge, Mass.).

**Schlichting, H. An approximate method for calculation of the laminar boundary layer with suction for bodies of arbitrary shape.** Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1216**, 85 pp. (1949).

[Translated from Aerodynamisches Institut der Technischen Hochschule Braunschweig, Ber. 43/13 (1943).] Except for minor differences, this paper is identical with another paper of the author [Ing.-Arch. **16**, 201–220 (1948); these Rev. **10**, 411]. *F. E. Marble* (Pasadena, Calif.).

**Schlichting, H. Lecture series "Boundary Layer Theory." I. Laminar flows.** Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1217**, iv+165 pp. (1949).

[These lectures were originally published in German by the Zentrale für Wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters, Berlin-Adlershof, 1942.] A fairly full account of what was known in 1942 on the laminar viscous flow of an incompressible fluid. *M. J. Lighthill* (Manchester).

**Schlichting, H. Lecture series "Boundary Layer Theory." II. Turbulent flows.** Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1218**, i+136 pp. (1949).

[Cf. the preceding review.] An account of what was known in 1942 on turbulence, with special reference to the boundary layer, together with a theory of the origin of turbulence (through instability of the laminar boundary layer) which was original at the time the lectures were given. *M. J. Lighthill* (Manchester).

**Brun, Edmond, et Vasseur, Marcel. Quelques définitions concernant la couche limite.** J. Recherches Centre Nat. Recherche Sci. **2**, 118–120 (1948).

**Brun, Edmond, et Vasseur, Marcel. Contribution à l'étude thermique de la couche limite laminaire.** J. Recherches Centre Nat. Recherche Sci. **2**, 121–126 (1948).

The first of these two papers presents definitions and physical interpretations of several characteristic lengths defined in terms of the density, velocity and temperature distributions of the boundary layer. Some of these characteristic lengths arise in the second paper which deals with the two-dimensional boundary layer in a heat-conducting compressible fluid where the fluid density $\rho$, the viscosity $\mu$, and the thermal conductivity $\lambda$ may be considered functions

of the temperature only. Consequently the analysis holds only for moderate fluid velocities where the variation of density with pressure may be neglected.

The authors propose an iteration solution such that in the first approximation $\rho$, $\mu$, $\lambda$ are considered constant and in the $n$th approximation the density, viscosity, and thermal conductivity distributions used would be those calculated from the temperature distribution of the $(n-1)$th approximation. The analysis is restricted to the first approximation inasmuch as the thermal properties are of primary interest. The first approximation velocity distribution is independent of the temperature distribution and is identical with that of the boundary layer in an incompressible fluid. The temperature distribution is then described by the energy equation using the known velocity components for the incompressible boundary layer. When the free stream velocity distribution is prescribed to be $U\sim x^m$, $m=\beta/(2-\beta)$, where $\beta$ is arbitrary and $x$ is the distance along the surface measured from the forward stagnation point, then the differential equation for the temperature distribution $\theta(\eta, x)$ is

$$xf'\frac{\partial\theta}{\partial x}-\frac{f}{2-\beta}\frac{\partial\theta}{\partial\eta}-\frac{\lambda}{(2-\beta)\mu c_p}\frac{\partial^2\theta}{\partial\eta^2}=\frac{U^2}{c_p(2-\beta)}(f''^2-\beta f'),$$

where $c_p$ is the specific heat of the fluid at constant pressure, $\eta$ is the dimensionless distance measured normal to the surface: $\eta=y\{\frac{1}{2}(m+1)\rho U/\mu x\}^{\frac{1}{2}}$, and $f(\eta)$ is the dimensionless form of the stream function of the incompressible boundary layer. The values of $f(\eta)$ have been determined by Hartree [Proc. Cambridge Philos. Soc. **33**, 223–239 (1937)] for $-0.198\leqq\beta\leqq1.60$. By calling $\theta(\eta, x)=x^{2m}\Theta_1(\eta)+\Theta_2(\eta, x)$ the ordinary differential equation

$$2\beta f'\Theta_1-f\Theta_1'-\frac{\lambda}{c_p\mu}\Theta_1''=\frac{U^2}{2c_p}(2f''^2-2\beta f'),$$

and the homogeneous partial differential equation

$$(2-\beta)xf'\partial\Theta_2/\partial x-f\partial\Theta_2/\partial\eta-(\lambda/c_p\mu)\partial^2\Theta_2/\partial\eta^2=0$$

are to be satisfied.

The authors consider the problem of (1) determining the temperature distribution of the wall so that no heat is transferred from the boundary layer and (2) finding the rate of heat transfer when the wall temperature is prescribed. Each of these problems is reduced to the numerical integration of an ordinary differential equation. The authors discuss briefly the possible solution of problems where the free stream velocity is prescribed arbitrarily.

*F. E. Marble* (Pasadena, Calif.).

**Coleman, W. S. Analysis of the turbulent boundary layer for adverse pressure gradients involving separation.** Quart. Appl. Math. **5**, 182–216 (1947).

The author proposes a method of calculating the development of a turbulent boundary layer and compares it with the partly empirical methods of von Doenhoff-Tetervin and of Garner. The present method is still based on certain assumptions regarding the conditions inside the boundary layer. The parameters occurring in the present scheme agree with those in Garner's equation, but the discrepancies between the three methods are not important in the final results. The author emphasizes the value of the present discussion in establishing the essential parameters, and states that "for ease and rapidity of calculation in practical cases, the empirical approach may prove more attractive."

*C. C. Lin* (Cambridge, Mass.).

**Frenkiel, F. N. The decay of isotropic turbulence.** J. Appl. Mech. **15**, 311–321 (1948).

The law of decay of isotropic turbulence is discussed in the cases of large and small Reynolds numbers of turbulence by using the von Kármán-Howarth equations and applying Loitsiansky's theorem. Then the turbulence level decreases respectively according to the inverse five-sevenths and the inverse five-fourths power law, with a suitably chosen hypothetical origin of time. Power laws of change of scale also follow. The author determines the origin of time by using experimental data. Comparisons with experiments are made, indicating better agreement in the law of decay than in the change of scale. The difference is attributed largely to the lack of isotropy.

*C. C. Lin* (Cambridge, Mass.).

**Hamel, Georg. Eine gemässigt turbulente Bewegung. (Abhandlungen zur Hydrodynamik. XII.)** Z. Angew. Math. Mech. **28**, 65–72 (1948).

The author considers a "turbulent" motion satisfying the following conditions: (1) it is a two-dimensional stationary motion of an incompressible viscous fluid in the region $-\infty<x<+\infty$, $-h\leqq y\leqq h$; (2) the velocity components and the vorticity remain finite; (3) the $x$-component of velocity is nowhere zero. He then introduces a "turbulent function" which has the constant boundary value unity and is larger than unity elsewhere except when the flow is laminar. The "turbulence function" and the velocity distribution are related to each other. The relation between this work and (time-dependent) turbulent motion in the usual sense is not discussed.

*C. C. Lin.*

**Ledinegg, M. Der Mechanismus der Turbulenz.** Österreich. Ing.-Arch. **2**, 244–260 (1948).

The author tries to develop a theory of turbulence by examining the behavior of the individual vortices. The concepts and arguments used are, however, more suggestive than rigorous. After introducing a few assumptions, he derives the velocity distribution for the center part of the turbulent flow through a circular pipe.

*C. C. Lin.*

**Obuhov, A. M. The pulsation of pressure in a turbulent flow.** Doklady Akad. Nauk SSSR (N.S.) **66**, 17–20 (1949). (Russian)

In this paper the author presents a sufficiently general method for the determination of the structure function of the pressure field in an isotropic turbulent flow of incompressible fluid. Under the assumption of statistical homogeneity and isotropy of the field of velocities and the pressure field, and under the further assumption that the fourth moments of the derivatives of the field of velocities have approximately the same relation to the second moments as in normal distributions [cf. Millionščikov, Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1941**, 433–446; these Rev. **4**, 121], the equation

$$\Delta^2\Pi(\rho)=-\sigma^2\frac{\partial^2 D^{\alpha\beta}}{\partial\xi^\alpha\partial\xi^\gamma}\cdot\frac{\partial^2 D^{\alpha\gamma}}{\partial\xi^\beta\partial\xi^\delta}$$

connecting the structure function of the pressure field $\Pi(\rho)=\overline{(p(M_2)-p(M_1))^2}$ and the structure function of the field of velocities $D^{\alpha\beta}(\xi)=\overline{(v^\alpha(M_2)-v^\alpha(M_1))(v^\beta(M_2)-v^\beta(M_1))}$ is set up. The structure tensor for the locally isotropic field of velocities [cf. Kolmogoroff, C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 301–305 (1941); these Rev. **2**, 327] has

the form

$$D^{\alpha\beta}(\xi)=D_{nn}(\rho)\delta^{\alpha\beta}+\left(\frac{D_{nn}-D_{tt}}{\rho^2}\right)\xi^\alpha\xi^\beta,$$

where $D_{nn}$ and $D_{tt}$ are the longitudinal and the transversal structure functions of the field of velocities, $\rho$ is the distance between the points $M_1$ and $M_2$, and $\sigma$ is the density of the medium. A particular case, $D_{tt}=b^2\rho^{\frac{1}{3}}$, $D_{nn}=\frac{4}{3}b^2\rho^{\frac{1}{3}}$ ($b$ a constant) is considered. This leads to the relation $\Pi(\rho)=\sigma^2D_{11}^2(\rho)$ [cf. Kolmogoroff, loc. cit.; C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 16–18 (1941); these Rev. **3**, 221; Obuhov, C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 19–21 (1941); Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1941**, 453–466; these Rev. **3**, 221; **4**, 121]. The author points out that the pulsation of pressure in a turbulent flow apparently is connected with the problem of interaction between the field of sound and the turbulence; however, from the acoustical point of view of a turbulent flow, not only the spatial structure, but also the time spectrum of the pulsation of pressure may be of interest.

*E. Leimanis* (Vancouver, B. C.).

**Thomas, T. Y. The consistency relations for shock waves.** J. Math. Physics **28**, 62–90 (1949).

This is a continuation of two previous papers by the author [same J. **26**, 62–68 (1947); **27**, 279–297 (1949); these Rev. **8**, 611; **10**, 494]. The situation considered is that of a pointed symmetric obstacle in a uniform supersonic stream with an attached curved shock. The oncoming flow has Mach number $\mu$ and makes an angle $\alpha(P)$ with the shock direction at the point $P$ of the shock. Let $s$ be arc length along the shock line, $\eta$ arc length along any streamline behind the shock; $k$ and $K$ are the curvatures of the shock and of the streamline at the shock. The author then shows that under appropriate conditions of differentiability there exist "consistency relations" of the form

$$d^nK/d\eta^n=G_n(M,\alpha)d^nk/ds^n+H_n$$

for $n=0, 1, 2, \cdots$, where $H_n$ is a polynomial in $k$ and its derivatives of order less than $n$. Here $G_n$ and the coefficients of the terms of $H_n$ are rational functions of $M^2$ whose coefficients are functions of $\alpha$. These relations are particularly interesting at the vertex $V$ of the obstacle, where the left members are determined by the geometry of the obstacle. Then there will in general exist for a given $M$ "singular" values of $\alpha$ (the initial shock direction) for which some $G_n(M,\alpha)$ vanishes, and for which the corresponding $k_n$ is consequently indeterminate.

In the first of the two papers cited above, the first derivatives of the hydrodynamical variables were evaluated behind the shock, and from these an explicit expression for $G_0$ was derived ($H_0$ being zero). In the second, this function $G_0(M,\alpha)$ was tabulated and graphed against $\alpha$ for various values of $M$. In the present paper, the second derivatives are obtained, and the function $G_1(M,\alpha)$ explicitly presented. Tables and graphs are also given, exhibiting the singular values of $\alpha$ (two in number) for each $M$ considered.

*D. P. Ling* (Murray Hill, N. J.).

**Thomas, T. Y. On conditions for steady plane flow with shock waves.** J. Math. Physics **28**, 91–98 (1949).

In steady two-dimensional supersonic flow past a pointed symmetric obstacle there are consistency relations which hold along an attached shock [see the preceding review]. These take the form $d^nK/d\eta^n=G_n(M,\alpha)d^nk/ds^n+H_n$, where $K$ and $k$ are the curvatures of streamline and shock at the shock, $\eta$ and $s$ are arc lengths along streamline and shock, and the $G_n$ are rational functions of $M^2$ (free flow Mach number) with coefficients depending on $\alpha$, the shock inclination angle. In what follows these relations are taken at the vertex of the obstacle. A value of $\alpha$ for which any $G_n$ vanishes for fixed $M>1$ is called "singular relative to $M$." The author considers a time dependent shock line, with unsteady flow behind it, uniform flow in front. He then shows that a sufficient condition that the shock and the flow behind it be steady is that $\alpha$ be nonsingular relative to $M$.

*D. P. Ling* (Murray Hill, N. J.).

***Brinkley, Stuart R., Jr., and Kirkwood, John G. Theory of the propagation of shock waves from cylindrical charges of explosive.** Proc. Symposia Appl. Math., Vol. I, pp. 48–54. American Mathematical Society, New York, N. Y., 1949. $5.25.

The authors consider explosion waves from cylindrical charges when (a) the explosion proceeds radially from the center-line of the charge and (b) the explosion wave travels axially down the charge. Using the Euler equations of motion and continuity, the Rankine-Hugoniot shock equations, the energy equation (which allows for entropy change in the medium as the shock passes) and an arbitrary equation of state, the exact differential equations are formulated. The approximation used consists in replacing the energy-time relationship, which actually depends on integrals of the equations, by an explicit approximation which is adequate for most practical purposes. When this is done in case (a) there results a pair of ordinary differential equations for peak pressure and energy at the shock as a function of the radial coordinate $R$. The equations involve functions which can be tabulated from the exact thermodynamical properties of the medium. An essentially similar result is given for case (b). Here the solution is again a function of time and the radial coordinate only, since the radial and axial coordinates are connected by the equation of the surface of revolution formed by the shock wave in the exterior medium.

*D. P. Ling* (Murray Hill, N. J.).

***Polachek, H., and Seeger, R. J. On shock-wave phenomena: Interaction of shock waves in gases.** Proc. Symposia Appl. Math., Vol. I, pp. 119–144. American Mathematical Society, New York, N. Y., 1949. $5.25.

The present paper constitutes a detailed summary of the existing mathematical theory concerning the interaction of plane shock waves in gases. The subject is treated economically but in detail and a brief review can do no more than suggest the contents. The Rankine-Hugoniot relations are first set up in a number of useful forms as the basis for the later material. Then the "simple" theory is set forth, i.e., the theory which assumes that the regions between the shocks, or between shocks and rigid boundaries, are regions of constant pressure. The cases of two-shock reflections ("regular" reflection) and three-shock (Mach or Y) reflection are fully treated. In the latter case a plane density discontinuity is involved. Recent experiments indicate a considerable discrepancy between theory and observation, and the authors develop a natural modification of the simple theory for both cases mentioned above. This consists in allowing a Prandtl-Meyer expansion to replace the constant pressure hypothesis in all regions where such an expansion wave is possible.

A third investigation is made of the refraction of shock waves at the intersection of two media of differing proper-

ties. One would expect a transmitted wave and a reflected wave. When the reflection is a shock, a general type of three-shock configuration results. The reflected wave may be a rarefaction wave, and a different configuration results. The complexity of the algebraic equations suggests a study of special cases to orient the investigator. The cases of normal incidence of a shock, and arbitrary incidence of an acoustic wave are accordingly studied. Special attention is given to the problems of an "impedance match," where the reflected wave is absent, and of total reflection. The final section contains some general results on multi-shock configurations. Tables and graphs are given wherever appropriate, and an excellent bibliography is appended.

*D. P. Ling* (Murray Hill, N. J.).

**Starr, Victor P. A mathematical theory of convection.** J. Meteorol. **6**, 188–192 (1949).

The author's summary is as follows. Since the motions of the atmosphere involve the presence of external heat sources and heat sinks, it is important to incorporate these effects into the hydrodynamical theory of air motions. In the discussion contained in this paper a method for accomplishing such a synthesis of hydrodynamics and thermodynamics is formulated for the case of two-dimensional motion. The use of the method is illustrated by an exact integration of the equations for the case of a circular vortex. Possible further elaborations of the method are discussed with a view of throwing light upon the fundamental nature of the general circulation of the atmosphere.

*H. A. Panofsky* (New York, N. Y.).

**Fox, E. N. The diffraction of two-dimensional sound pulses incident on an infinite uniform slit in a perfectly reflecting screen.** Philos. Trans. Roy. Soc. London. Ser. A. **242**, 1–32 (1949).

In a previous paper [same Trans. Ser. A. **241**, 71–103 (1948); these Rev. **10**, 166], the author obtained solutions for the normal incidence of a pulse field (two-dimensional) on a perfectly reflecting infinite strip or half plane. The author now outlines a method whereby the results of his first paper could be used to derive solutions for two-dimensional diffraction problems involving strips or half-planes as obstacles. An application is made to the problem of an infinite screen containing an infinite slit of uniform width upon which is incident a known pulse field. As a special case, the author considers a plane sharp-fronted pulse of constant pressure incident normally upon such a screen. The method and the results are rather too detailed to be described here in full. *A. E. Heins* (Pittsburgh, Pa.).

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## *Elasticity, Plasticity*

**Richter, Hans. Verzerrungstensor, Verzerrungsdeviator und Spannungstensor bei endlichen Formänderungen.** Z. Angew. Math. Mech. **29**, 65–75 (1949). (German. Russian summary)

To define strain in a continuous medium which undergoes a finite deformation, the author starts with the matrix $A$ which represents the mapping of a neighborhood of a point $\xi$ in the undeformed medium on to a neighborhood of the corresponding point $x$ in the deformed medium: $dx = A d\xi$. In a plastic material the history of deformation is important and, hence, the knowledge of $A$ alone is not sufficient. For an elastic material, on the other hand, $A$ completely characterizes the deformation. For an anisotropic elastic material, the rigid body rotation contained in $A$ is important, and $A$ itself must be used to describe the deformation. For an isotropic elastic material, however, this rigid body rotation is unessential; the strain tensor is then obtained by eliminating this rigid body rotation in a suitable manner. The author proceeds to establish postulates which should be satisfied by any acceptable definition of the strain tensor $S$. First of all, it must be possible to build up this tensor from the elements of the matrix $A$. Secondly, the tensor should not be influenced by a rigid body rotation which precedes the deformation characterized by the matrix $A$. Thirdly, if $A_1$ and $A_2$ denote pure deformations with coincident principal axes, $S_1 = S(A_1)$ and $S_2 = S(A_2)$ the corresponding strain tensors and $S_0 = S(A_1 A_2)$ the strain tensor corresponding to the deformation characterized by $A_1 A_2 (= A_2 A_1)$, there should exist a monotonic function $f(S)$ such that $f(S_1) + f(S_2) = f(S_0)$. Finally, the definition of the strain tensor should reduce to the customary one when infinitesimal deformations are considered. The author introduces a logarithmic strain tensor and shows that it satisfies these postulates. *W. Prager* (Providence, R. I.).

***Kaplan, Wilfred. Numerical methods in the solution of problems of nonlinear elasticity.** Proc. Symposia Appl. Math., Vol. I, pp. 194–196. American Mathematical Society, New York, N. Y., 1949. $5.25.

A discussion is given of the methods of relaxation and gradients for determining the equilibrium position of a body from the solution of a set of simultaneous nonlinear equations. A compromise method is suggested for such problems. No discussion is given of the convergence of the solution in those nonlinear cases where the equilibrium position is unstable. *S. Levy* (Washington, D. C.).

**Symonds, P. S. On the general equations of problems of axial symmetry in the theory of plasticity.** Quart. Appl. Math. **6**, 448–452 (1949).

It has been previously found that axially symmetric plasticity problems, in which the yield condition is taken to be that for which two principal stress differences equal the yield stress in pure tension, reduce to statically determinate problems. The method of characteristics can then be employed. The present paper investigates the question of whether the same problems with the Mises flow condition lead to hyperbolic differential equations. The equilibrium conditions, yield condition, stress-strain laws (of the Saint-Venant, Lévy, Mises flow theory), and the compatibility equations are set up in terms of the stress-deviations and displacements in cylindrical coordinates. From these, the equations of the characteristic curves can be determined. It is found that the characteristics are real if and only if the problem is one of plane strain. Thus the equations are hyperbolic if and only if the axially symmetric case reduces to plane strain. *G. H. Handelman* (Pittsburgh, Pa.).

**Symonds, P. S. The determination of stresses in plastic regions in problems of plane flow.** J. Appl. Phys. **20**, 107–112 (1949).

The author gives a numerical-graphical procedure for integrating the hyperbolic differential equations governing the plastic stress distributions in plane flow. The author's method is particularly applicable to determining the stresses adjacent to a boundary which is part of a plastic zone.

Three alternative procedures for determining the shear lines or characteristics are given (tangent, mean value, and radius of curvature). The second method is illustrated for an elliptic free boundary which can, within certain restrictions, be used to determine the yield load of a bar with elliptic notches. Related problems are discussed.

*G. H. Handelman* (Pittsburgh, Pa.).

***Drucker, D. C. Stress-strain relations for strain hardening materials: Discussion and proposed experiments.** Proc. Symposia Appl. Math., Vol. I, pp. 181–187. American Mathematical Society, New York, N. Y., 1949. $5.25.

The difference between "deformation" and "flow" theories of plasticity is discussed. In the former the state of strain is determined completely by the state of stress; in the latter strain increments are determined by the stress and the stress increments, so that the total strain is a cumulative effect depending on the path of loading. It is suggested that tests involving a stress system which is rotated without change of magnitude would distinguish between the two types of theory most simply. Means of carrying out and interpreting such tests are discussed. *E. H. Lee.*

**Trifan, D. A new theory of plastic flow.** Quart. Appl. Math. 7, 201–211 (1949).

A new stress-strain relation for an isotropic, incompressible strain-hardening material exhibiting a gradual transition from the elastic to the plastic state is given in the form $s_{ij}^* = 2G\epsilon_{ij}^* - p(E)E^*\epsilon_{ij}$, where $\epsilon_{ij}$ is the strain tensor, $s_{ij}$ the stress deviation, $E$ the strain intensity, and * denotes rate of change with respect to a parameter $t$. The function $p(E)$ can be determined from a tensile test. It is shown that, for a given state of strain throughout the body and given velocities on the surface such that the surface integral of the normal velocity component vanishes, the rate of change of strain is uniquely determined throughout the body. A minimum principle for loading, together with certain extensions, is given. Through the use of a perturbation scheme (odd powers), this theory is compared with a theory of plastic deformation, $s_{ij} = 2G'(E)\epsilon_{ij}$. The linear and third order terms are found to agree for both theories. Conditions are also given for agreement of higher order terms.

*G. H. Handelman* (Pittsburgh, Pa.).

**Trifan, D. On the plastic bending of circular plates under uniform transverse loads.** Quart. Appl. Math. 6, 417–427 (1949).

The author compares the bending of a circular plate in the plastic range under uniform transverse loads for the two cases in which the material follows a stress-strain law of the plastic deformation type (strain determined by stress) and a stress-strain law of the stress-theory of plastic flow (stress rate determined by strain and strain rate). Both materials are assumed to be homogeneous and isotropic and to exhibit gradual transition from the elastic to the plastic range. The flow theory used is the same as that given by the author [see the preceding review].

The basic equations for the small deflections of a thin plate are derived and expressed in a perturbation scheme. The perturbation equations are solved for the special case of a clamped circular plate subject to a uniformly distributed lateral load. Perturbation equations are also worked out for the same plate in which the material follows a deformation theory. Curves for the radial and circumferential stresses in the upper surface of the plate are given in both cases. The results, although differing considerably from those of the elastic theory, are almost identical for both cases. A curve for the nondimensional load versus the nondimensional work is also given. *G. H. Handelman.*

**Torre, C. Der Spannungszustand in einem schweren Erdkörper.** Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 156, 583–592 (1948).

A half-space is assumed to be filled with uniform grains of approximately uniform size subject to gravity but without cohesive forces. The plane boundary may form any angle with the direction of gravity. Such a situation would be approximated by a thick deposit of loose sand which everywhere had attained the angle of frictional repose under the overlying uniform load. The author applies the method of envelopes, which he assumes to be of paraboloidal form, to determine the stress distribution and the families of slip surfaces under "plastic" flow. In view of the assumed symmetry, he restricts his analysis to two dimensions. He refers to his previous article [Österreich. Ing.-Arch. 1, 36–50 (1946); 316–342 (1947); these Rev. 8, 115; 9, 395]. The results are presented in tabular form. *J. B. Macelwane.*

**Sadowsky, M. A., and Sternberg, E. Stress concentration around a triaxial ellipsoidal cavity.** J. Appl. Mech. 16, 149–157 (1949).

The exact solution is given in closed form for the elastic stresses produced by an ellipsoidal cavity in an infinite isotropic medium under constant stress at infinity. In this problem the cavity is free of surface traction and the axes of the ellipsoid are parallel to the directions of arbitrary principal stress at infinity. Displacement potentials are employed to find five linearly independent solutions for which all the stress components are tabulated in ellipsoidal coordinates. They are combined to nullify the tractions on the cavity due to the uniform fields. The interesting stress values for uniaxial tension are given in graphical form for various shape ratios taking Poisson's ratio $\nu$ as 0.3. Maximum stresses are not greatly influenced by $\nu$.

*D. C. Drucker* (Providence, R. I.).

**Stowell, Elbridge Z. Critical shear stress of an infinitely long plate in the plastic region.** Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1681, 19 pp. (1948).

The problem stated in the title is developed as a special case of the general theory previously given by the author [same Tech. Notes, no. 1556 (1948); these Rev. 10, 82]. It is assumed that no unloading takes place during buckling, that is, the plate remains wholly in the plastic range. It is also assumed that the plate is infinitely long with side edges elastically restrained against rotation. The applied shear stresses are uniformly distributed. The general equation of energy balance (work done equals strain energy in the plate plus strain energy of the restraints) is applied to the case of shear buckling. A system of oblique coordinates, with wave angle $\phi$, is introduced. The deflection function for the elastic case, with the half wave length so adjusted as to make the critical stress a minimum, is used to get an approximate value for the critical stress $\tau$ in the plastic range. The wave angle $\phi$ is also chosen to make $\tau$ a minimum. The ratio $\eta$ of the critical stress in the plastic range to that of the elastic range is computed as a function of the secant modulus, tangent modulus, and edge restraint. It is found that $\eta$ is the same for the two extreme cases of simply supported

edges and clamped edges. The author also shows that Gerard's secant modulus [J. Appl. Mech. 15, 7–12 (1948)] applies when the plasticity coefficient for the material is constant. *G. H. Handelman* (Pittsburgh, Pa.).

***Friedrichs, K. O. The edge effect in the bending of plates.** Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 197–210. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

The solution of the differential equation governing the deflection of a thin plate does not satisfy, in general, the three boundary conditions of prescribed shear forces, twisting moments, and bending moments. The author studies this "edge effect" by a systematic expansion of the problem with respect to powers of the thickness of the plate. This approach is sufficiently general to allow even the surface tractions, rather than just the resultants, to assume arbitrarily prescribed values. The discussion is restricted to the neighborhood of a completely free edge, and a semi-circular plate is considered for simplicity. The main purpose of the investigation is the description of the deviation of the actual stress and displacement values near the edge from those of the Kirchhoff (classical) theory.

The "excess quantities," the differences between the actual and the induced boundary values, change quickly from the edge to very nearly zero some distance from the edge. These are studied by an expansion process similar to that employed in discussing boundary layer phenomena. It is stated that the excess stresses can be expressed in terms of two potential functions and a biharmonic function. The manner in which the excess stresses die down on moving away from the edge is investigated by studying the asymptotic representations of these auxiliary functions. It is shown that, at a distance from the edge greater than the thickness of the plate, the excess stresses are negligible. It is also stated that, except for a correction due to curvature of the boundary and a slight deviation in numerical factors, these results agree well with those of E. Reissner [J. Math. Physics 23, 184–191 (1944); J. Appl. Mech. 12, A-69–A-77 (1945); these Rev. 6, 195; 7, 42]. A more detailed analysis of this problem is promised in the future. *G. H. Handelman.*

***Friedrichs, K. O. The edge effect in bending and buckling with large deflections.** Proc. Symposia Appl. Math., Vol. I, pp. 188–193. American Mathematical Society, New York, N. Y., 1949. $5.25.

The author presents a concise survey of a number of boundary layer theory problems in linear and nonlinear elasticity. Among the problems which are discussed by means of their relevant differential equations are those concerning (1) linear bending of a flat plate under initial tension, (2) nonlinear bending and buckling of circular plates, and (3) nonlinear membrane theory of spherical shells. The problem of linear bending of spherical shells, the effective width problem in the theory of buckling of flat plates, and the problem of the nonlinear buckling theory of spherical shells are also discussed. The author emphasizes the importance of the mathematical problems concerning appropriate boundary conditions which arise in connection with the reduction in order of a system of linear and especially nonlinear differential equations describing a given physical situation. Such reduction is often possible if the existence of a boundary layer is given or discovered.

*E. Reissner* (Cambridge, Mass.).

**Greenberg, H. J., and Prager, W. Direct determination of bending and twisting moments in thin elastic plates.** Amer. J. Math. 70, 749–763 (1948).

A method for finding direct approximations for the bending and twisting moments in thin elastic plates is developed in a manner analogous to that found for the general theory of elasticity by W. Prager and J. L. Synge [Quart. Appl. Math. 5, 241–269 (1947); these Rev. 10, 81]. The equation of equilibrium relating moments and applied normal pressure plays the role of the general equilibrium equations, the curvatures are analogous to the strains, the deflection to the displacements, and the moments to the stresses. Thus it is possible to define a state of stress in function space and to show that the distribution of bending and twisting moments which satisfies equilibrium, compatibility and boundary conditions is represented by a vector which lies on a certain hypercircle. The center and radius of this hypercircle can be given in terms of approximate solutions. The authors give an iteration scheme for obtaining the best hypercircle from a given number of approximate states. The method is applied to a square clamped plate under a uniform normal load. Five iterations have been carried out and it has been found that the bending moments at the corner and center agree to within 5% of those calculated by Nadai [Die elastischen Platten, Springer, Berlin, 1925, p. 184]. *G. H. Handelman* (Pittsburgh, Pa.).

***Levy, Samuel. Large deflection theory for rectangular plates.** Proc. Symposia Appl. Math., Vol. I, pp. 197–210. American Mathematical Society, New York, N. Y., 1949. $5.25.

The author presents a complete and clearly written survey of work by himself and collaborators on the subject of finite transverse deflections of thin elastic plates. To be solved are two well-known simultaneous nonlinear partial differential equations subject to appropriate boundary conditions. The author's approach to these problems is through the use of trigonometric double series expansions for Airy's stress function $F$ and the transverse deflection $W$. These series satisfy the boundary conditions of simply supported edges. The case of built-in edges is reduced to the foregoing by introducing into the theory certain specific forms of surface loads which are equivalent to applied edge moments. The author points out that, while this procedure seems to work well, it would be good to have a proof that his use of divergent series is legitimate (as it undoubtedly is). The trigonometric series reduce the differential equations of the problem to an infinite system of simultaneous cubic equations for the coefficients of the series. This system is solved by setting all coefficients except the first few equal to zero and by solving the resultant finite nonlinear system by iteration. It is observed that more terms have to be taken in the linear part of the equations than in the nonlinear part, for satisfactory accuracy. Explicit results thus obtained for various conditions of loading and support are presented in a series of diagrams. The paper concludes with specific recommendations for additional investigations. To the list of references given in this paper, one could add papers by G. Schnadel, K. Marguerre, C. T. Wang, and others who obtained numerical results, similar to a few of those here, using different methods of solution of the same system of differential equations.

*E. Reissner* (Cambridge, Mass.).

**Cox, H. L. The distortion of a flat rectangular plate in its own plane.** Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2200 (8822), 37 pp. (1948).

Using the principle of least total energy (which is usually confined to problems of elastic stability), the author obtains expressions for the edge displacements of a flat rectangular plate having any system of loading applied to the edges. The effect of members attached to the edges of the plate is also considered. It is shown that, apart from uniform dilation, the displacements may be resolved into four components: (a) shearing parallel to the edges, (b) shearing at 45° to the edges, (c) bending about a neutral axis parallel to one pair of edges and (d) bending about a neutral axis parallel to the other pair of edges. Bending and pure shearing are then considered as special cases of the solution and the usual formulas are obtained. In an appendix, the method is applied to obtain the distribution of axial stress around a rectangular hole in a long tube of circular cross-section when the tube is bent. *H. D. Conway* (Ithaca, N. Y.).

**Seth, B. R. Bending of an elliptic plate with a confocal hole.** Quart. J. Mech. Appl. Math. 2, 177–181 (1949).

Using typical solutions in elliptic coordinates of the differential equation $\nabla^4\omega = p/D$, the author obtains an approximate solution to the problem of the bending of a thin elliptic plate with a confocal hole. The plate is subjected to a uniform pressure and the edges are clamped.

*H. D. Conway* (Ithaca, N. Y.).

**Kuhn, R. Beitrag zur Berechnung der Halbkreisplatte.** Ing.-Arch. 12, 307–319 (1941).

The author tabulates the deflection, moments, shearing forces, and boundary reactions for a simply supported semicircular plate with uniform normal loading. These results, obtained from the first three terms of the Fourier series solution, are compared with those obtained from difference equations using a radial network. The deflections from the latter analysis are approximately an average of five percent smaller than from the more exact analysis. The moments show a greater average error of eight or ten percent when obtained from difference deflections by difference equations and only three percent when obtained from exact deflections by difference equations. Some comparisons are made when $\sigma = 0$ and $\sigma = \frac{1}{4}$. *D. L. Holl* (Ames, Iowa).

**Reissner, Eric. On bending of curved thin-walled tubes.** Proc. Nat. Acad. Sci. U. S. A. 35, 204–208 (1949).

The bending of a curved thin-walled tube is considered as a problem of determining an axi-symmetric stress distribution in a shell of revolution. The results of H. Reissner and E. Meissner on the bending and stretching of shells of revolution are modified by not requiring that the displacements be single-valued and rotationally symmetric. The author takes the circumferential displacement $v$ to be of the form $v = kr\theta$, where $r$ and $\theta$ are polar coordinates and $k$ is a constant. The displacement-strain, compatibility, equilibrium, and stress-strain equations for an orthotropic tube are then developed. These reduce to two simultaneous, second order differential equations. After these are solved, a relation between $k$ and the applied bending moment can be found. The special cases of von Kármán, Golovin, and Winkler and Resal are obtained from these equations through a suitable choice of parameters.

*G. H. Handelman* (Pittsburgh, Pa.).

**Vlasov, V. Z. Computation of thin-walled prismatic shells.** Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1234, 51 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 8, 361–394 (1944); these Rev. 7, 142.

**Michielsen, Herman F. The behavior of thin cylindrical shells after buckling under axial compression.** J. Aeronaut. Sci. 15, 738–744 (1948).

This paper is a continuation of the work by von Kármán and Tsien [same J. 8, 303–312 (1941); these Rev. 4, 63] in which the buckling of cylindrical shells was studied by considering nonlinear large deflections. The author derives simplified relations between the critical stress and strain and the wave length parameters. The final equilibrium positions are found from the condition that the total energy is stationary with respect to the wave length parameters. Curves are given for the critical stresses as well as formulas for the stresses in the median surface and the bending stresses. *G. H. Handelman* (Pittsburgh, Pa.).

**Signorini, Antonio. Sulle dighe a volta non cilindrica.** Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 185 = Pont. Acad. Sci. Comment. 10, 375–406 (1947).

Après quelques considerations d'ordre pratique sur les difficultés qui s'opposent à une étude mathématiquement complète de la stabilité des digues en ferrobéton, l'auteur accepte la schématization proposée par F. Tölke dans son livre [Talsperren und Staumauern, Springer, Berlin, 1938]. Il observe cependant que l'analyse mathématique de Tölke, qui réduit le problème à la résolution d'un problème au contour pour une seule équation aux dérivées partielles du quatrième ordre doit être considérée insuffisante. L'auteur démontre qu'en effet on doit résoudre le système formé par une telle équation linéaire du quatrième ordre et une équation différentielle ordinaire, aussi linéaire du troisième ordre, moyennant la condition de satisfaire à quelques limitations géométriques qu'on peut facilement admettre dans la pratique. *B. Levi* (Rosario).

**Swida, W. Die Formänderungsenergiesätze in Anwendung auf den elastisch-plastischen Zustand.** Ing.-Arch. 16, 221–230 (1948).

The author is concerned with energy theorems associated with the elastic-plastic bending of a beam consisting of a non-strain-hardening material which obeys Hooke's law up to the yield point. Elastic unloading is assumed. From the simple graph of concentrated transverse load $P$ versus displacement $\delta$ of its point of application, he deduces the following relations: $dU/d\delta = P$, $dU_k/dP = \delta$, $dU_e/dP = \delta_e$ and $dU_r/dP = \delta_r$, where $U$ is the work done, $U_e$ is the elastic energy (recoverable upon unloading), $U_k = U_e + U_r$ is what the author calls "potential energy," $U_r$ is the nonrecoverable part of $U_k$ and is called the residual or permanent energy of deformation, $\delta_r$ is the permanent displacement, $\delta_e$ is the elastic displacement. The author notes that the first two relations are not new but fails to identify the quantity $U_k$ as being the familiar "complementary energy" of F. Engesser [Z. Architektur u. Ingenieurwesen Hannover 35, 734–742 (1889)]. The last two relations are new. The author applies some of these relations to examples involving statically determinate beams and derives the principle of virtual displacements for use in statically indeterminate problems. Finally he notes that the Maxwell-Betti reciprocity theorem is no longer valid in the elastic-plastic

domain. All of these things are well known in more general terms. *H. J. Greenberg* (Pittsburgh, Pa.).

**Chen, Pei-Ping. The equivalent loading method and the equivalent beam method.** Quart. Appl. Math. **7**, 183–200 (1949).

Three-dimensional prismatic frameworks treated in a previous paper by the author [J. Franklin Inst. **238**, 325–334 (1944); these Rev. **6**, 74] have been extended to include nonuniform cross sections. The equivalent loading method, developed for nonuniform loading, is now supplemented with the equivalent beam method for variable cross sections. These methods amount to a type of symmetrisation resulting from the relaxation of the loading conditions and the cross sectional form. A number of illustrative examples are given. *B. R. Seth* (Ames, Iowa).

**Cotugno, Nicoletta, e Mengotti-Marzolla, Clelia. Approssimazione per eccesso della più bassa frequenza di una piastra ellittica omogenea incastrata.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 324–326 (1948).

The calculation is made according to the Ritz method. *P. Funk* (Vienna).

**Young, Dana. Vibration of a beam with concentrated mass, spring, and dashpot.** J. Appl. Mech. **15**, 65–72 (1948).

The author discusses the problem of determining the natural frequencies of transverse vibration of a uniform beam having the indicated loading and any type of end supports. The following superposition procedure is stated to be quite general. Consider that the beam, together with its several loads, will vibrate in harmonic motion $y=W(x)H(\omega, t)$ and that the force acting between each load and its point of attachment will also be harmonic: $f=F(x)H(\omega, t)$. These expressions, containing the sought-for parameter $\omega$, may be inserted in the standard (small amplitude theory) equation for a vibrating beam, i.e., $EI\partial^4y/\partial x^4+(m/l)\partial^2y/\partial t^2=f$, resulting in an ordinary equation which may be solved by the method of characteristics for the displacement at the point of application of the forcing term $f$ (equivalent to the load). This solution, expressing displacement in terms of the parameter $\omega$, may be equated to the necessarily equal displacement obtained as a steady-state solution to the ordinary differential equation for the load alone, acted upon by the forcing term $-f$, and the composite expression solved graphically for the desired natural frequencies of the system.

Three examples are treated: (1) cantilever beam plus mass and spring located at a point $h$; (2) cantilever beam plus dashpot located at $h$; and (3) cantilever beam plus mass, spring, and dashpot located at $h$, this last producing an awkward expression in $\omega$ not discussed by the author. Case (2) is compared to the results of McBride [Trans. A.S.M.E. **65**, A168–A172 (1943)]. *J. H. Bigelow.*

**Dörr, J. Das Schwingungsverhalten eines federnd gebetteten, unendlich langen Balkens.** Ing.-Arch. **16**, 287–298 (1948).

Under the usual assumptions of small elastic bending, the author treats the case of a concentrated transverse load travelling with constant velocity along an infinitely long uniform beam resting on an elastic foundation. Damping forces are presumed to act on the beam which are proportional to the local deflection velocity. The author restricts attention to the steady-state response of the beam system, which remains after the initial transients have died away. By a straightforward application of Fourier integral theory, it is shown how the steady-state response can be derived directly, without the need to study the initial transient motion. By introducing certain simplifications, such as the assumption of small damping, approximate and relatively simple solutions for the problem are deduced. It is found that a critical velocity of the travelling load exists, for which large beam responses are encountered; for this circumstance, the energy input to the system on the part of the travelling load varies inversely as the square root of the damping coefficient. For speeds below this critical speed, the energy input by the travelling load varies directly with the damping coefficient. For speeds higher than the critical speed, it is found that, to the first order in the damping coefficient, the energy input is independent of the value of the damping coefficient. The author points out the analogy between this behavior and that of the flow of a compressible fluid about an obstacle (the critical speed is analogous to the sonic velocity). The paper concludes with a brief discussion of the manner in which the complete solution for the problem, including the transient motion, can be deduced. *M. Goland* (Kansas City, Mo.).

**Petrašen', G. The two-dimensional problem of Lamb for an infinite elastic layer bounded by parallel planes.** Doklady Akad. Nauk SSSR (N.S.) **64**, 783–786 (1949). (Russian)

L'auteur étudie le problème de Lamb pour une couche élastique infinie limitée par deux plans parallèles, en supposant qu'une de ces surfaces ne subit aucun effort et que l'effort sur l'autre surface limite ne dépend que d'une seule variable. En supposant que l'action intense perturbatrice est quasi-ponctuelle dans l'espace et le temps on peut effectivement calculer la propagation des ondes élastiques qui en résultent. L'application de la méthode de séparation incomplète des variables permet d'en déduire toutes les conséquences physiques qui peuvent intéresser le physicien ou le seismologue. *V. A. Kostitzin* (Paris).

**Zvolinskiĭ, N. V. Asymptotic development of the solution of the problem of the propagation of a disturbance from a point source in an elastic half-space covered by a layer of compressible fluid.** Doklady Akad. Nauk SSSR (N.S.) **65**, 145–148 (1949). (Russian)

L'auteur suppose qu'une couche fluide compressible d'épaisseur $H$ couvre un demi-espace élastique. Au moment $t=0$ entre en action une source ponctuelle momentanée très intense disposée quelque part dans le demi-espace. Le processus de propagation de la perturbation peut être représenté par la somme de trois fonctions. La mise en équations du problème et leur résolution ont été données par l'auteur dans une note précédent [mêmes Doklady (N.S.) **59**, 1081–1084 (1948); ces Rev. **9**, 482]. Dans cette note-ci l'auteur donne une autre forme de solution, commode pour étudier l'allure asymptotique de ces fonctions. *V. A. Kostitzin* (Paris).

**Fridman, M. M. The diffraction of a plane elastic wave by a semi-infinite rectilinear slit, free of stress.** Doklady Akad. Nauk SSSR (N.S.) **66**, 21–24 (1949). (Russian)

Dans un travail précédent l'auteur a étudié le problème à deux dimensions de la diffraction d'une onde élastique plane

par une fente rectilinéaire semi-infinie, à liaisons dures [mêmes Doklady (N.S.) 60, 1145–1148 (1948); ces Rev. 9, 636]. Dans cet article l'auteur étudie le même problème pour une fente libre de tout effort. Il en donne une solution effective.

*V. A. Kostitzin* (Paris).

## MATHEMATICAL PHYSICS

### *Optics, Electromagnetic Theory*

**Franz, Walter. Zur Theorie der Beugung.** Z. Physik 125, 563–596 (1949).

A method for approximately solving acoustical and optical diffraction problems is indicated, Kirchhoff's solution for the black screen being obtained as a first-order approximation. A novel interpretation of Kirchhoff's formula permits the author to develop higher-order approximations in a significant manner. It is not required that the screen be black, nor that the diffracting obstacle be large compared to the wavelength. The author's results for some typical cases are checked against the corresponding exact results.

*C. J. Bouwkamp* (Eindhoven).

**Bouwkamp, C. J. On the transmission coefficient of a circular aperture.** Physical Rev. (2) 75, 1608 (1949).

Levine and Schwinger [same Rev. (2) 74, 958–974 (1948); these Rev. 10, 221] developed a variational principle for solving the problem of diffraction of a scalar plane wave by an aperture in an infinite plane screen. For a circular aperture, and a plane wave of normal incidence, they provided a transmission coefficient which differed in the third term from that of the author [Groningen dissertation, 1941; these Rev. 8, 179]. The author points out that he has made a numerical error and gives his corrected result. Details of an approximation method are promised elsewhere.

*A. E. Heins* (Pittsburgh, Pa.).

**Levine, Harold, and Schwinger, Julian. On the transmission coefficient of a circular aperture.** Physical Rev. (2) 75, 1608–1609 (1949).

The authors point out that their transmission coefficient [reference in the preceding review] is in numerical error from the third term onward. The corrected form now agrees with Bouwkamp's corrected one [see the preceding review].

*A. E. Heins* (Pittsburgh, Pa.).

**Levine, Harold, and Schwinger, Julian. On the theory of diffraction by an aperture in an infinite plane screen. II.** Physical Rev. (2) 75, 1423–1432 (1949).

[See the two preceding reviews for a reference to part I.] The technique applied is almost identical with that of part I. The main difference is that now the wave function is expressed in terms of the discontinuity in its normal derivative at the screen, and not in terms of the aperture field. An integral equation is derived and a corresponding variational principle is set up. For the circular aperture and normal incidence, numerical results are compared with those obtained by the first variational principle and with the available exact results. *C. J. Bouwkamp* (Eindhoven).

**Kisun'ko, G. V. Variational principles for boundary (diffraction) problems of electrodynamics.** Doklady Akad. Nauk SSSR (N.S.) 66, 863–866 (1949). (Russian)

A variational formulation with attendant use of the semi-direct method of approximation is applied to electrodynamic problems of the diffraction type where a primary field is excited by prescribed current and/or charge distributions in a space $V_1$ with given dielectric constant, etc., closed partly by ideal conductors and partly by a surface (aperture) $S$, across which it communicates with a similarly constituted space $V_2$; either or both may stretch to infinity, in which case the radiation condition supplants the boundary conditions. The tangential electric and magnetic components $\mathfrak{E}$, $\mathfrak{H}$ along $S$ are sought.

By writing the field distributions in $V_1$, $V_2$ as the superposition of two excitations: (1) the original sources and (2) the unknown $\mathfrak{E}$, $\mathfrak{H}$, the continuity conditions across $S$ lead to two formally corresponding nonhomogeneous linear operational equations for $\mathfrak{E}$ and $\mathfrak{H}$. These are replaced by functionals involving the latter as well as $\mathfrak{E}(\mathfrak{H})$ and integrated over $S$; vanishing of the variation of such an integral under variation of all nonfixed (nonimposed) field quantities is then equivalent to fulfillment of the operational equations. This leads to approximation by the semi-direct method through variation of the adjustable coefficients of the linear sets of "trial" functions; if, in particular, the first parameter represents a normalizing multiplier, the nonhomogeneous linear operator on $\mathfrak{E}(\mathfrak{H})$, on the basis of the chosen functional, becomes equal to zero.

*H. G. Baerwald* (Cleveland, Ohio).

**Mirimanov, R. G. On a new method for the solution of problems on the reflection of electromagnetic waves from thin unclosed surfaces of finite curvature.** Doklady Akad. Nauk SSSR (N.S.) 66, 641–644 (1949). (Russian)

In the region exterior to a given homogeneous body the scalar potential of a field with given angular frequency $\omega=kc$ must satisfy a certain integral equation in which the kernel $e^{ik|\mathfrak{r}-\mathfrak{r}'|}/|\mathfrak{r}-\mathfrak{r}'|$ appears. The body is supposed to be of uniform thickness so small that the effect of the lateral surface is negligible. By making use of the boundary conditions, and expanding the potential at an internal point in Taylor's series containing its value and derivatives at a surface point on the same normal, a method is obtained for solving in successive approximations, with the thickness as the small parameter. The zeroth approximation is taken to be the potential of a given point source. A similar calculation can be made with components of the vector potential or the Hertzian potential. The method makes no use of the assumption of ideal conductivity, and employs the exact boundary conditions. On the other hand, it depends on the rapid convergence of the Taylor series, and this is not considered by the author. In many important cases, such as those involving metal reflectors, suitable convergence is not to be expected. *W. H. Furry* (Cambridge, Mass.).

**Kogan, S. H. The propagation of waves along an infinite spiral.** Doklady Akad. Nauk SSSR (N.S.) 66, 867–870 (1949). (Russian)

The propagation of electromagnetic waves along an infinite spiral leads, under the approximating assumptions that the pitch ratio as well as the ratios of wire radius to spiral radius, step and wavelength are all very small, to a simple integral equation; this is solved by a Fourier expansion whose $n$th term represents the $n$th mode of propagation, the radial dependence of which is described by cylinder

functions of $n$th order. The propagation constants of the different modes may be real or imaginary, as in those of wave guides. The solution derived is used for numerical computation of specific examples to be compared with measurements made by J. Kraus and J. Williamson [J. Appl. Phys. 19, 87–96 (1948)], and good agreement is obtained. *H. G. Baerwald* (Cleveland, Ohio).

**Meerovič, Ê. A. An operational method applied to the equations of electrodynamics in a bounded three-dimensional region.** Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1949**, 186–191 (1949). (Russian)

It is pointed out that the solution of Adler is erroneous [J. Appl. Phys. **16**, 545–550 (1945); these Rev. **7**, 99]. Adler obtains for the Fourier transform of the derivative of a function the expression

$$F[\varphi'(x)]=(j\pi n/l)F[\varphi(x)]+(-1)^n[\varphi(l)-\varphi(-l)].$$

He gets rid of the boundary values by assuming that the function, defined for $0<x<l$, is continued for $-l<x<0$ in an even way: $\varphi(x)=\varphi(-x)$. The boundary values cancel, but the solution obtained in this way does not take cognizance of these boundary values, and this is erroneous. In applying this method to Maxwell's equation, Adler continues the functions $E$ and $H$ differently in the various equations. As a consequence the transformed equations cannot be used as a system of simultaneous equations, and the results are of no value for computational purposes. In conclusion the author solves the problem of a prismoidal cavity resonator, with perfectly conducting boundaries, taking care that each component appears in each of the equations in a similar way, and obtains for the transform of the electrical vector the expression

$$\bar{E}(n,p)=(\bar{n}'\times\mu\bar{H}(n,0)+\mu\epsilon p\bar{E}(n,0))/(\epsilon\mu p^2-n'^2),$$

which defines each harmonic in terms of its boundary values. An expression in terms of the $j$-variable is also given. The oscillations are undamped since no losses are considered. *M. Daniloff* (Cambridge, Mass.).

**Kleinwächter, Hans. Die Darstellung der ebenen Bewegung von Elektronen in magnetischen und elektrischen Feldern mit komplexen Ortsvektoren.** Arch. Elektrotechnik **33**, 479–486 (1939).

The classical Lorentz equations of motion for a charged particle moving in a plane are written in terms of the complex number $z=x+iy$, where $x$ and $y$ are the coordinates of the position of the particle. When this is done the equations can be handled very simply in a great variety of cases. The problem of the motion of three bodies whose mutual attraction is proportional to the distance between them is handled in a similar manner. *A. H. Taub.*

**Zin, G. Su alcune questioni di elettrodinamica classica relative al moto dell'elettrone.** Nuovo Cimento (9) **6**, 1–23 (1949).

C. J. Eliezer [Rev. Modern Physics **19**, 147–184 (1947); these Rev. **9**, 69] suggested that the Dirac equations for the motion of an electron in an electromagnetic field should be modified by introducing the advanced as well as the retarded potentials in the radiation terms. This introduces a factor of $2k+1$ in the differential equations (Dirac's equations correspond to $k=0$). The present paper is a critical discussion of Eliezer's equation for the case of the rectilinear motion of an electron of charge $e$ in the field of a stationary particle of charge $ne$. Four cases are considered, corresponding to $2k+1$ greater than or less than 0 and $n$ greater than or less than 0, and it is shown that the two particles may collide if $n(2k+1)>0$ but not if $n(2k+1)<0$. Thus if $n<0$ we must also have $2k+1<0$ to obtain results which are physically acceptable.

However, consideration of the nonrelativistic equations of an elementary oscillator (a charged particle subject to an elastic force) shows that the assumption $2k+1<0$ leads to results which are at variance with known experimental facts. Thus the author concludes that Eliezer's suggested modification cannot be justified. *M. C. Gray.*

**Pignedoli, Antonio. Sul problema delle aurore polari. Moto di un corpuscolo elettrizzato in presenza di un dipolo magnetico e in prossimità di uno dei poli, essendo l'altro polo molto lontano.** Atti Sem. Mat. Fis. Univ. Modena **1**, 17–49 (1947).

The problem of the motion of an electrified particle in the neighborhood of a single magnetic pole can be solved by quadratures. The author outlines this known solution and then uses a perturbation method to obtain an approximate solution for the motion of the particle in the neighborhood of one pole of a magnetic dipole of large but finite length $2a$. The paper is almost entirely algebraic and the resulting formulas are too complicated to be reproduced here. *M. C. Gray* (Murray Hill, N. J.).

**Bulgakov, A. A. On the geometry of electromagnetic systems.** Doklady Akad. Nauk SSSR (N.S.) **66**, 63–65 (1949). (Russian)

The voltages and currents in a system of $n$ fixed magnetically coupled coils are interpreted geometrically in terms of the motion of a point in a configuration space. *E. N. Gilbert* (Murray Hill, N. J.).

**Zelyah, Ê. V. Foundations of a general theory of linear four-poles.** Avtomatika i Telemehanika **10**, 135–148 (1949). (Russian)

The four-poles considered are networks containing linear passive elements, independent sources (voltage generators), and sources with outputs depending linearly on the branch currents and node voltages within the network. The various linear relationships among the currents $I_1$, $I_2$ and voltages $U_1$, $U_2$ at the two accessible terminal-pairs are tabulated. For example, $U=ZI+E$, where $Z$ is a (nonsymmetric) impedance matrix and $E$ consists of the voltages appearing at the terminals when the network is open circuited. *E. N. Gilbert* (Murray Hill, N. J.).

**Stommel, Henry. The theory of the electric field induced in deep ocean currents.** J. Marine Research **7**, 386–392 (1948).

The problem of the induced electric field due to an arbitrary motion of a conducting ocean in the earth's magnetic field is formulated. On the assumption that the magnetic field is constant, it is shown that a scalar potential for the electric field exists and satisfies Poisson's equation $\nabla^2\phi=\mathbf{H}\cdot(\nabla\times\mathbf{V})$, where $\mathbf{H}$ is the magnetic field, and $\mathbf{V}$ is the velocity vector field, both known. Assuming a nonconducting bottom and a sinusoidal, horizontal velocity distribution in an upper layer of the ocean and zero velocity in a lower layer, an illustrative solution is given. The effect of conductivity of the ocean bottom is considered, and it is concluded that it may generally be neglected for shallow currents in the deep ocean, but not necessarily for currents in shallow channels. The author promises future results for

more realistic velocity profiles to be obtained by relaxation methods and to be compared with actual measurements.

*J. W. Miles* (Los Angeles, Calif.).

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### *Quantum Mechanics*

**Jardetzky, W. Ein Satz von Ruđer Bošković und die Grundlagen der Quantentheorie.** Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. **7**, 31–42 (1941).

This paper relates a modified inverse square law formulated in 1758 to Bohr's atomic model of 1913.

*P. Franklin* (Cambridge, Mass.).

**Fedorov, F. I. On the solution of relativistic wave equations.** Doklady Akad. Nauk SSSR (N.S.) **65**, 813–814 (1949). (Russian)

The use of perturbation methods to obtain the solution $\psi$ of $(H+U)\psi=0$ from the solution $\psi_0$ of $H\psi_0=0$ depends on the application of the operator $H^{-1}$. The author considers the case of relativistic equations for which $H=\gamma^k\nabla_k+\kappa$, where $\gamma^k$ denotes a set of four matrices. By means of a type of algebraic relation given by Bhabha [Rev. Modern Physics **17**, 200–216 (1945); these Rev. **7**, 272], which every such relativistic operator must satisfy, the operator $H^{-1}$ is reduced to the application of well-known Green's functions together with a finite number of differentiations and matrix multiplications. The application to scattering of light by electrons and by particles of spins 0 and 1 is stated briefly.

*W. H. Furry* (Cambridge, Mass.).

**Kallmann, Hartmut, und Päsler, Max. Behandlung des Oszillators und der Diracschen Gleichungen.** Ann. Physik (6) **4**, 46–56 (1948).

Continuation of two earlier papers [same Ann. (6) **2**, 292–304, 305–320 (1948); these Rev. **10**, 378].

**Kallmann, Hartmut, und Päsler, Max. Ergänzende Bemerkung zu unserer Arbeit "Allgemeine Behandlung des $H$-Atoms mit beliebigen Anfangsbedingungen. . . ."** Ann. Physik (6) **4**, 90–91 (1948).

The paper in question is the second cited in the preceding review.

**Becker, R., und Leibfried, G. Zur Methode der Quantisierung von Wellengleichungen.** Z. Physik **125**, 347–358 (1949).

A new method of quantization is described which, however, produces the same end results as the usual method. Starting from the Hamiltonian function, the procedure regards the observables, whose commutation relations are assumed known, as operators in a Hilbert space which is to be explicitly constructed. A transformation is sought which relates these to operators obeying a standard form of commutation relation. This insures that the eigenvalues are nonnegative integers. In terms of these operators the Hilbert space is constructed. The process is illustrated for the case of the linear harmonic oscillator and for the Schrödinger-Gordon equation for a real Lorentz-invariant function. A sketch of this method has appeared previously [Physical Rev. (2) **69**, 34 (1946); these Rev. **7**, 404]. *O. Frink.*

**Broer, L. J. F. On the existence and uniqueness of the solution of the fundamental equation in the theory of metallic conduction.** Nederl. Akad. Wetensch., Proc. **50**, 885–891 = Indagationes Math. **9**, 396–402 (1947).

Nach der Elektronentheorie der Metalle muss das Lorentz-Blochsche Gleichungssystem

$$(\partial f_k/\partial t)_{\text{feld}}-\sum_{k'} I_{kk'}+\sum_{k'} I_{k'k}=0$$

gleichzeitig mit der Bedingung $\sum f_k=N$ bestehen (wo $f_k$ die Fermische Verteilungsfunktion und $N$ die Zahl der Elektronen ist und die obigen Gleichungen zum Ausdruck bringen, dass sich $f_k$ in der Zeit nur infolge von Quantenübergänge ändern kann). Die Frage ob dieses Gleichungssystem nur eine Lösung besitzt, ist das sogenannte dritte Problem der Elektronentheorie der Metalle, die vom Verfasser hier das erste mal exakt bewiesen wird. Zuerst wird gezeigt, dass die Übergangswahrscheinlichkeiten vier Bedingungen erfüllen, unter denen auch der Ergodensatz [bewiesen in der Quantenmechanik von J. von Neumann, Z. Physik **57**, 30–70 (1929)] vorkommt. Nach der Herleitung von zwei Hilfsätzen wird das Problem zuerst für den feldfreien Fall bewiesen und dann auf den Fall verallgemeinert, dass $f$ und die Übergangswahrscheinlichkeiten nach der Feldintensität in eine Reihe entwickelt werden können. Der Beweis würde nur an der Konvergenzgrenze versagen, doch treten bei hohen Feldstärken sowieso schon Ionisationserscheinungen und der Tuneleffekt auf.

*T. Neugebauer* (Budapest).

## BIBLIOGRAPHICAL NOTES

**Atti del Seminario Matematico e Fisico dell'Università di Modena.**

Volumes 1 and 2 appeared in 1947 and 1948, respectively.

**Bulletin de la Société des Mathématiciens et Physiciens de la R.P. de Serbie.**

Volume 1, no. 1 appeared in 1949. The main title is in Serbian, Vesnik Društva Matematičara i Fizičara Narodne Republike Srbije. There is also a subtitle in Russian. The contents are in Serbian, usually with French and Russian summaries. The journal is published in Belgrade.

**Kōdai Mathematical Seminar Reports.**

Volume 1, number 1 appeared in 1949; there are expected to be six issues a year. This is a publication of the Department of Mathematics of the Tokyo Institute of Technology.

**Proceedings of Symposia in Applied Mathematics.**

The first two volumes were published in 1949 by the American Mathematical Society. They contain the papers presented at the Society's Symposia on Non-Linear Problems in Mechanics of Continua and on Electromagnetic Theory, held in 1947 and 1948, respectively. The papers will be reviewed separately in Mathematical Reviews.

V. 2 was not published until 1950.

**Osaka Mathematical Journal.**

Volume 1, no. 1 appeared in 1949. This is a publication of the Department of Mathematics, Osaka University.

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